Index volatility

Principles of the skew

The ISMA Centre’s Carol Alexander shows how principal component analysis should be used on the fixed strike deviations from at-the-money volatility

Many financial markets are characterised by a high degree of collinearity. It occurs when there are only a few important sources of information in the data, which are common to many variables. In this case, principal component analysis (PCA) is a standard method for extracting the most important uncorrelated sources of variation in a multivariate system.

PCA is not just about term structures of interest rates or futures, although most readers will be familiar with the method in this context. The standard interpretation of the first component as the trend, the second component as the tilt and the third component as the curvature holds for any highly correlated ordered system, not just a term structure. So when implied volatilities are ordered by strike or maturity, PCA application will reveal the standard trend-till-curvature interpretation of the first three principal components.

Several principal component models of volatility smiles and skews have been based on daily changes in implied volatilities, by strike and/or by moneyness. Derman and Kamal (1997) analysed S&P500 and Nikkei 225 index options, where the daily change in the volatility surface is specified by delta and maturity. Skladopulos, Hodges and Clewlow (1998) applied PCA to first differences of implied volatilities for fixed maturity buckets, across both strike and moneyness metrics. And Fengler et al (2000) employed a common PCA that allows options on equities in the Dax of different maturities to be analysed simultaneously.

There is an important difference between the research just cited and the approach taken in this paper. Instead of applying PCA to daily changes in implied volatilities, a PCA is applied to daily changes in the deviations of fixed strike volatilities from at-the-money (ATM) volatility. The advantages of this approach are both empirical and theoretical. On the empirical front, time series data on fixed strike or fixed delta volatilities often display much negative autocorrelation, possibly because markets over-react. But the daily variations in fixed strike deviations from ATM volatility are much less noisy than the daily changes in fixed strike (or fixed delta) volatilities. Consequently the application of PCA to fixed strike deviations from ATM volatility yields more robust and intuitive results.

The theoretical advantage of this approach is that the models of the skew in equity markets that were introduced by Derman (1999) can be expressed in a form where fixed strike volatility deviations from ATM volatility always have the same relationship with the underlying index. The particular market regime is determined only by a different behaviour in ATM volatility. Thus the stability of PCA on fixed strike deviations from ATM volatility is implied by Derman’s models.

Derman (1999) asked: how should implied volatilities be changed as an equity index moves? He developed three models that are described in box 1. In each model there will be a parallel shift in all volatilities as the index moves, where the size of this shift is determined by the current market regime. The model presented here extends Derman’s models to allow non-parallel shifts in the skew as the index moves. It uses PCA to actually quantify the sensitivities of implied volatilities to changes in the underlying price.

The model has applications to all types of im-

Box 1. ‘Sticky’ models

Derman (1999) formulates three different regimes of price-volatility behaviour in equity index markets. A different linear parameterisation of the volatility skew applies in each regime, and each parameterisation implies a different type of stickiness for the local volatility in a binomial tree. For this reason, the skew parameterisations are known as Derman’s ‘sticky’ models.

Denote by $\sigma_{0}(t)$ the implied volatility of an option, with maturity $t$ and strike $K$, $\sigma_{ATM}(t)$ the volatility of the $t$-maturity ATM option, $S$ the current value of the index and $\sigma_{0}$ and $S_{0}$ the initial implied volatility and price used to calibrate the tree:

a) In a range bounded market, skews should be parameterised as:

$\sigma_{c}(t) = \sigma_{0} - b(t)(K - S)$

b) In a stable trending market, skews should be parameterised as:

$\sigma_{c}(t) = \sigma_{0} - b(t)(K - S)$

So fixed strike volatility $\sigma_{c}(t)$ is independent of the index level and $\sigma_{ATM}$ will decrease as index increases:

$\sigma_{ATM}(t) = \sigma_{0} - b(t)(S - S_{0})$

So fixed strike volatility $\sigma_{c}(t)$ will increase with the index level. But $\sigma_{ATM}(t)$ will be independent of the index, since $\sigma_{ATM}(t) = \sigma_{0}$

c) In jumpy markets, skews should be parameterised as:

$\sigma_{c}(t) = \sigma_{0} - b(t)(K + S) + 2b(t)S_{0}$

$\sigma_{ATM}(t) = \sigma_{0} - 2b(t)(S - S_{0})$

The ATM volatility will also decrease as the index goes up and increase as the index falls, twice as fast as the fixed strike volatilities.

The range-bounded model (a) is called the ‘sticky strike’ model because local volatilities will be constant with respect to strikes. That is, each option has its own binomial tree, with a constant volatility that is determined by the strike of the option. For a given change in the index, the root of the tree is moved to the new level of the index. The same tree is still used to price the option.

The trending markets model (b) is called the ‘sticky delta’ model because local volatilities are constant with respect to the moneyness (or equivalently the delta) of the option. That is, it is the moneyness of the option that determines the (still constant) local volatility in the tree. As the index changes, the delta of the option also changes and we consequently move to a different tree, the one corresponding to the current option delta.

In the ‘sticky tree’ model (c), the local volatilities are no longer constant. There is, however one unique tree that can be used to price all options determined by the current skew. This is the implied tree described in Derman and Kani (1994).
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1a. Parallel shift in skew deviations as price moves up

1b. Non-Parallel shift in skew deviations as price moves up

Fixed strike deviations from ATM volatility

The relationship between fixed strike deviations from ATM volatility and the underlying price is the same in all of Derman’s ‘sticky’ models. In fact, for any maturity τ there will be a linear relationship between the deviation of a fixed strike volatility from ATM volatility and the underlying price given by:

\[ \sigma_K(t) - \sigma_{ATM}(t) = -b(t)(K - S) \]  

(1)

For any given maturity, the deviations of all fixed strike volatilities from at-the-money volatility will change by the same amount b(t) as the index level changes, as shown in figure 1a. Four strikes are marked on this figure: a low strike \( K_1 \), an initial ATM strike \( K_2 \), a new ATM strike \( K_3 \) after the index level moves up \( K_2 \) and a high strike \( K_4 \). In each of the three market regimes, the range of the skew between \( K_2 \) and \( K_3 \), that is \( \sigma_2 - \sigma_1 \), will be the same after the index move. Thus as the underlying price moves, the fixed strike volatilities will shift in parallel, and the range of the skew will remain constant. The direction of the movement in fixed strike volatilities depends on the relationship between the original ATM volatility \( \sigma_1 \) and the new ATM volatility \( \sigma_2 \):

- In a range-bounded market, \( \sigma_2 = \sigma_1 - b(t) \), but fixed strike volatilities have all increased by the same amount b(t), so a static scenario for the skew by strike should be applied;
- When the market is stable and trending, \( \sigma_2 = \sigma_1 \) and there is an upward shift of b(t) in all fixed strike volatilities;
- In a jumpy market, \( \sigma_2 = \sigma_1 - 2b(t) \), so a parallel shift downward of b(t) in the skew by strike should be applied.

These observations suggest that a method for testing whether Derman’s ‘sticky’ models are supported by empirical evidence is to perform a PCA of the daily change in fixed strike volatility deviations from ATM volatility, denoted \( \Delta \sigma_K - \sigma_{ATM} \). Equation (1) implies that only the first principal component should be significant, but if it is found that the second or higher principal components are significant factors for determining movements in \( \Delta \sigma_K - \sigma_{ATM} \), then the parallel shifts in the skew that are implied by the ‘sticky’ models will not apply.

PCA of skew deviations

The model has been developed on one-month, two-month and three-month implied volatilities for European options of all strikes on the FTSE 100 index for the period January 4 1998 and March 31 1999. The implied volatility surfaces, including currency option smiles and swaption skews. But the present paper focuses on its application to the skew in the FTSE 100 between January 4 1998 and March 31 1999. It is found that the sensitivity of a fixed strike volatility to movements in the index can change according to market conditions. In particular, for short term volatility the range of the skew (the difference between low strike volatility and high strike volatility) will normally fluctuate over time.

The PCA of fixed strike deviations is based on the model:

\[ \Delta \sigma_K(t) - \sigma_{ATM}(t) = \omega_{K, 1}(t)P_1(t) \]

\[ + \omega_{K, 2}(t)P_2(t) + \omega_{K, 3}(t)P_3(t) \]  

(2)

where \( t \) is the volatility maturity and \( K \) is the strike of the volatility, both fixed. Daily data on \( \Delta \sigma_K(t) - \sigma_{ATM}(t) \) is used to estimate the principal components \( P_1(t), P_2(t) \) and \( P_3(t) \), and the factor weights \( \omega_{K, 1}(t), \omega_{K, 2}(t) \) and \( \omega_{K, 3}(t) \). A PCA of \( \Delta \sigma_K(t) - \sigma_{ATM}(t) \) for a fixed maturity \( t \) gives excellent results (box 2). Alexander (2000) shows that for fixed maturity volatility skews in the FTSE 100 index option market during most of 1998, 80-90% of the total variation in skew deviations can be explained by just three key risk factors: parallel shifts, tilts and curvature changes. The parallel shift component accounted for around 65-80% of the variation, the tilt component explained a further 5 to 15% of the variation and the curvature component another 5% or so. The precise figures depend on the maturity of the volatility (one-month, two-month or three-month) and the exact period in time that the principal components were measured.

The immediate conclusion must be that the limitation of movements in volatility surfaces to parallel...
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Box 2. PCA for a fixed maturity

A

PCA for three-month implied volatility skew deviations gives the output in the tables shown. It is clear from table A that the first principal component only explains 74% of the movement in the volatility surface, and that the second principal component is rather important as it explains an additional 12% of the variation over the period. It is interesting that the factor weights shown in table B indicate the standard interpretation of the first three principal components in a term structure, as parallel shift, tilt and convexity components. Note that sparse trading in very out-of-the-money options implies that the extreme low strike volatilities show less correlation with the rest of the system, and this is reflected by lower factor weights on the first component.

B. Eigenvectors of correlation matrix

<table>
<thead>
<tr>
<th>Component</th>
<th>Eigenvalue</th>
<th>Cumulative $R^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>P1</td>
<td>13.3574</td>
<td>0.742078</td>
</tr>
<tr>
<td>P2</td>
<td>2.257596</td>
<td>0.8675</td>
</tr>
<tr>
<td>P3</td>
<td>0.691317</td>
<td>0.905906</td>
</tr>
</tbody>
</table>

3. Gamma estimates for one-month volatilities

The principal component representation equation (2) and the gamma coefficients in equation (3).

\[ \gamma(t) = \gamma_1(t) \Delta S \]

Thus the movement in fixed strike volatility deviations in response to movements in the underlying will be determined by the factor weights in the principal component only explains 74% of the movement in the volatility surface, and that the second principal component is rather important as it explains an additional 12% of the variation over the period. It is interesting that the factor weights shown in table B indicate the standard interpretation of the first three principal components in a term structure, as parallel shift, tilt and convexity components. Note that sparse trading in very out-of-the-money options implies that the extreme low strike volatilities show less correlation with the rest of the system, and this is reflected by lower factor weights on the first component.

When $\gamma_1$ is negative it is clear that $\delta_1$ will be less than $\delta_3$ and that $\gamma_2$ is normally a little greater than $\delta_1$, unless $\gamma_3$ is very large and negative. If $\gamma_2$ were extremely large and negative then $\delta_2$ would be less than $\delta_3$, but this never occurs empirically. On the other hand if $\gamma_2$ is positive then $\delta_2 > \delta_3$.

2When $\gamma_2$ is negative it is clear that $\delta_2$ will be less than $\delta_3$ and that $\gamma_1$ is normally a little greater than $\delta_3$, unless $\gamma_3$ is very large and positive. If $\gamma_1$ were extremely large and positive then $\delta_1$ would be greater than $\delta_3$, but this never occurs empirically. On the other hand if $\gamma_1$ is positive then $\delta_1 > \delta_3$.

Exponentially weighted moving average estimates (with $\lambda = 0.94$) of $\gamma_1$, $\gamma_2$ and $\gamma_3$ for each of the one-month, two-month and three-month maturities have been calculated. The gamma estimates for the one-month volatilities are shown in figure 3. For reasons of space the two-month and three-month gamma estimates are not shown, but are given in Alexander (2000). In all cases, the estimates of $\gamma_1$ are strongly positive throughout. Since $\gamma_1$ captures the parallel shift component of movements in the skew, we can deduce that most of the movement in the skew at all maturities can be attributed to a parallel shift up when the index falls.

For the two-month and three-month maturities, the index seems to have little effect on the second and third principal components - the estimates of $\gamma_2$ and $\gamma_3$ are close to zero for almost all the sample period. There are a couple of negative $\gamma_2$ periods during the springs of 1998 and 1999, when the range of the skew will have narrowed as the index moved up and widened as it moved down. But this effect is not as pronounced as in the one-month skew. The rest of the time, and particularly during the crash period, it would be reasonable to apply parallel shift scenarios for fixed strike volatilities of two-month and three-month skews.

A slightly different picture emerges for the movement of the one-month skew (figure 3). The estimate of $\gamma_1$ is often negative, particularly during the spring of 1998 and the spring of 1999. At these times the range of the skew was clearly decreasing when the index rose and increasing when the index fell, an effect which is very evident in figure 2. But there are two notable periods, just before the beginning of the crash and during the market recovery, when the estimate of $\gamma_1$ was strongly positive and $\gamma_3$ was strongly negative. On July 14 1998, several days before the FTSE 100 price started to plummet, there was a dramatic increase in $\gamma_1$ and decrease in $\gamma_3$ so that $\gamma_1 > 0$ and $\gamma_3 < 0$. During this period, the range of the one-month skew narrowed as the index fell. Then between October 8-12 1998, the FTSE 100 jumped up 8% in 2 days’ trading, from 4803 to 5190. At the same time, $\gamma_1$ jumped up and $\gamma_3$ jumped down, so that again $\gamma_1 > 0$ and $\gamma_3 < 0$, and the range of the one-month skew widened as the index moved up. The narrowing of the range of the skew as the index fell, and the consequent widening again as the market recovered, was driven by unexpectedly big movements in high strike volatilities.

Scenario quantification for implied volatility surfaces

We assume for a fixed maturity of 1, 2 or 3 months that:

\[ \Delta \gamma_{ATM}(t) = \beta(t) \Delta S \]

To capture the dependence of $\beta$ on the current market regime, we estimate $\beta$ with an exponentially weighted moving average (EWMA) model. The EWMA model allows us to capture time-varying parameters by assigning more weight to recent data and less weight to older data. This is particularly useful in financial markets where parameters can change rapidly.

4Of course, if one uses equally weighted averages to estimate beta then it may appear to jump, but this would only be because of the 'ghost features' that are an artefact of pushing a constant parameter model into a time-varying framework.
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As expected, the sensitivity of ATM volatility to changes in the FTSE is greater in one-month options than in two-month options, which in turn have greater sensitivity than three-month options. There is a striking pattern in figure 4 and it is very clear indeed that the sensitivity of ATM volatility moves with the level of the index. It does not jump unless the index jumps.4

Combining equations (2), (3) and (4) yields:

\[
\Delta \sigma_k(t) = \beta_k(t) \Delta S
\]

where the sensitivity of the fixed strike volatility \(\sigma_k(t)\) to the index is given by:

\[
\beta_k(t) = \beta(t) + \sum \omega_k(t) \gamma(t)
\]

This completes the model for changes in all fixed strike volatilities as the index moves.

During the crash period the sensitivities of all volatilities are greater and the change in the 5875 strike volatility sensitivity is very pronounced. Before the crash, it ranged between -0.005 and -0.01, indicating an increase of between 0.5 and 1 basis points for every FTSE point decrease at the beginning of the crash the 5875 sensitivity increased to about 1.5bp, and since the FTSE fell by 1500 points during the crash, that corresponds to a 22.5% increase in 5875 volatility. Then at the height of the crash, between October 1-9, the 5875 sensitivity became increasing large and negative as the FTSE index reached a low of 4786 on October 5. On October 9, the 5875 sensitivity was an impressive 0.026, indicating a further 2.8bp increase in 5875 volatility would have occurred for every point off the FTSE at that time.

What is interesting about the 4675 volatility sensitivity is that it is often far greater (in absolute terms) than the high strike volatility sensitivities. So most of the movement will be coming from the low strikes as the range of the skew narrows when the index rises and widens when the index falls. The 4675 volatility approximately gains about 1 or 2bp for every point fall in the FTSE index during the period, although the sensitivity varies considerably. At the end of the data period it is extraordinarily large because the range narrowing of the skew was very considerable.

Conclusion

This paper has presented a new principal component model of fixed strike volatility deviations from ATM volatility. It is based on the observation that fixed strike volatility deviations from ATM volatility are independent of the market regime, and that the regime is determined only by different behaviour in ATM volatility. The model presented here actually quantifies the change that should be made to any given fixed strike volatility per unit change in the underlying, at any point in time. Empirical application of the model to the FTSE 100 index options has shown that two-month and three-month skews should normally be shifted parallel as the index moves, as predicted by Derman’s models. In the ‘range-bounded’ markets in the spring of 1998 and 1999, there was also some narrowing in the range of the skew as the index moved up (and widening as the index moved down). But the range narrowing of the skew in a range bounded market scenario is much more pronounced in the one-month skew.

For very short term volatility in equity markets, the empirical analysis has revealed two distinct regimes. In stable markets, the range of the one-month skew narrows as the index moves up and widens as the index moves down. Most of the movement is in low strike volatilities and high strike volatilities remain relatively static as the underlying moves. But there is a second regime in short-term volatilities that applies to the jumpy markets during a market crash and recovery period. In this regime, the high strike volatilities move much more than usual, and in the recovery period after the 1998 crash the one-month skew range actually widened as the index moved up.

The model presented has very general applications because it admits non-linear movements in the volatility smile as the underlying moves. PCA is shown to be a powerful analytical tool for the construction of scenarios of an implied volatility smile surface in different market regimes.

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