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Model-free price hedge ratios for homogeneous claims on tradable assets

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1. A plethora of models

A standard stylized fact in option theory is that the empirically observed 'smile' and 'skew' shapes in Black–Scholes implied volatilities contradict the assumptions of the Black–Scholes option pricing model. This has motivated an explosion of models with different asset price dynamics that aim to price and hedge exotic options consistently with the calibration to current market prices of simple European calls and puts. Three main strands of research have been developed in a prolific literature: stochastic volatility (e.g. Hull and White 1987, Stein and Stein 1991, Heston 1993, and many others) where the variance or volatility of the price process is stochastic; local volatility (Dupire 1994, Derman and Kani 1994, Rubinstein 1994) where volatility is a deterministic function of time and the asset price; and jump/Lévy models (Merton 1976, Naik 1993, Geman 2002) where jumps in the price or volatility or both are allowed. There are also the 'hybrid' models, which combine stochastic and local volatilities (Dupire 1996, Derman and Kani 1998, Hagan *et al*. 2002, Alexander and Nogueira 2004), stochastic volatility and jumps (Bates 1996, Bakshi *et al*. 1997, Andersen *et al*. 2002) or local volatility and jumps (Andersen and Andreasen 2000, Carr *et al*. 2004). For an extensive review of these models see Jackwerth (1999), Skiadopoulos (2001), Psychoyios *et al*. (2003), Bates (2003) and Cont and Tankov (2004).

Quants that understand how to implement even some of these models, and perhaps to derive new models of their own, are highly prized. If a better model allows traders to gain a few basis points on every deal, this more than justifies a six-figure salary and a similar bonus. Given such a variety of option pricing models numerous studies have attempted to identify the best pricing and the best hedging models (e.g. Bakshi *et al*. 1997, Dumas *et al*. 1998, Das and Sundaram 1999, Buraschi and Jackwerth 2001, Andersen *et al*. 2002, and others). Unsurprisingly, the answer largely depends on the application. It may be that a good pricing model turns out to be a bad hedging model, as observed by Wilkens (2005) for lognormal mixture models, or vice versa. But since hedging costs are factored into the price of most deals the choice of a model often depends on its ability to price and hedge simultaneously.

This article reviews some recent developments on the derivation of model-free hedge ratios, and in particular the work by Bates (2005) and its extension by Alexander and Nogueira (2007). Bates showed that if the price process is 'scale-invariant' and the price of an option on or before expiry is homogeneous of degree one in the underlying asset and exercise price, then the option delta and gamma are model-free and, in the case of vanilla options, related to the slope and curvature of the implied volatility smile. Alexander and Nogueira extended this result to any contingent claim priced under a scale-invariant process, provided only that its pay-off function is homogeneous of some degree in the price dimension. They also show how to verify whether a pricing model is scale-invariant without knowing the returns distribution and prove that all price hedge ratios—delta, gamma and higher order—will be model-free and may be derived from the prices of traded options.

These new theoretical results should precipitate a new approach to option hedging. They tell us that models can be classified so that all models that fall into the same category will have identical hedging properties.

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The purpose of this paper is to bring these new results to a wide audience.

2. Scale-invariant models

A price process is scale-invariant if and only if the marginal distribution of returns is independent of the price level. For instance, suppose the price process $S(t)$ can be described according to one of the two models below:†

$$\text{Model 1}: \quad \frac{\mathrm{d}S(t)}{S(t)} = \mu \mathrm{d}t + \alpha S(t)^{-1}\mathrm{d}B(t),$$

$$\text{Model 2}: \quad \frac{\mathrm{d}S(t)}{S(t)} = \mu \mathrm{d}t + \sigma\left(\frac{S(t)}{S(0)}\right)^{-1}\mathrm{d}B(t),$$

If we set the parameters so that α is numerically equal to $\sigma S(0)$, then both models will of course produce the same dynamics for the price process at any time $t > 0$. Yet, they are not equivalent since model 2 is scale-invariant whilst model 1 is not. To see why, consider first figure 1(a), which shows one simulated path for the price process according to either model 1 or 2. The parameters used here are $S(0) = 1$, $\mu = 0$, $\alpha = 15\%$, $\sigma = 15\%$ and we have simulated the price path over 1000 days. Note that the barrier, arbitrarily set to 1.10, has been hit in this particular simulation.

Now suppose the price suffers a 1-for-2 reverse split, so that the number of 'stocks' in the market are halved whilst the price of each stock doubles. This is equivalent to scaling the price process and everything else in the same dimension by two, i.e. $S(t) \mapsto 2S(t)$ and $B \mapsto 2B$, where B denotes the original level of the barrier.

Because the parameters α and σ are fixed the characteristics of the path produced by model 2 remain unchanged, yet the volatility in model 1 is halved! The simultaneous scaling of $S(0)$ and $S(t)$ implies that the diffusion term is dimensionless in model 2, but not in model 1. Figure 1(b) and (c) show the new paths after the reverse split. In figure 1(c) the path for model 2 is exactly the same as in figure 1(a) except for the y axis, which is scaled by two, and the barrier is hit at the same time as before. However in figure 1(b) the volatility becomes half of the original volatility and it turns out that the barrier is no longer hit. Thus the marginal distribution of returns in model 1 has changed as a result of the reverse split. For this reason, we say that model 2 is scale-invariant with respect to the price level, whilst model 1 is not.

Merton (1973) identified this 'constant returns to scale' property as a desirable feature for pricing options. This is not the same as a change of numeraire. The price of every asset in the economy changes if we change the numeraire, whilst scale invariance (also known as space homogeneity) refers only to a change in the unit for measuring the underlying price. If the probability

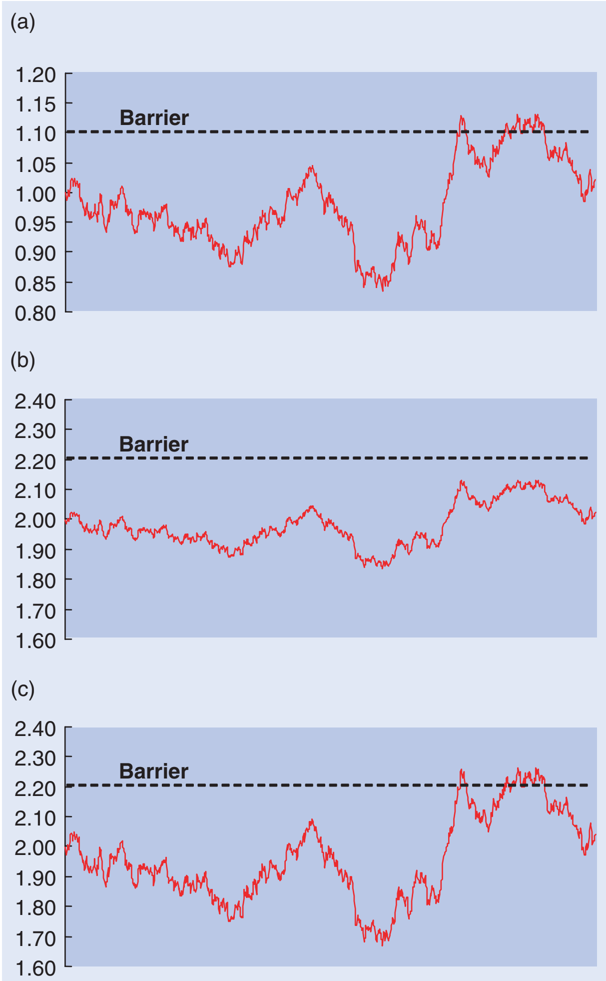

Figure 1. Simulation of the asset price path according to models 1 and 2 before and after the reverse split: (a) simulated path according to models 1 and 2 before the reverse split; (b) simulated path according to model 1 after the reverse split; (c) simulated path according to model 2 after the reverse split.

density of the underlying asset returns is scale-invariant, then the price of a standard American or European option will scale with the underlying price. Put another way, it does not matter whether the asset price is measured in dollars or in cents—the relative value of any option that has a pay-off function which is homogeneous of degree one will remain the same, as expected.

Hoogland and Neumann (2001) provide further motivation for scale invariance of price processes. Following their intuition, all tradable assets such as currencies, equities, equity indexes and commodities should be modelled with a scale-invariant process, whilst economic variables such as interest rates, volatility and inflation need not follow scale-invariant process.

For tradable assets, most stochastic volatility models are scale-invariant, even when interest rates are stochastic. Alexander and Nogueira (2007) show that a price process S is scale-invariant if it is a semi-martingale

†Model 1 is a variation of Cox's (1975) constant elasticity of variance (CEV) model. Model 2 is the scale-invariant version of model 1. The exponent, usually called β, has been set to -1 to induce a negative correlation between the asset price and the volatility.

and if relative price increments are dimensionless in the unit of measurement of S. So the process does not even need to be Markovian. Other broad classes of models are scale-invariant, including 'sticky-delta' local volatility models, mixture diffusions such as Brigo and Mercurio (2002), uncertain volatility models such as Avellaneda *et al.* (1995), volatility jump models such as Naik (1993) and GARCH models such as Nelson (1990) and Alexander and Lazar (2005). Even Lévy processes (Cont and Tankov 2004) are scale-invariant when the drift and Lévy density are dimensionless.

Almost all of the option pricing models in common use are scale-invariant. There are a few exceptions, including deterministic volatility models in which the instantaneous volatility is a static function of the asset price (e.g. Cox 1975, Dumas *et al.* 1998) and the 'sticky-tree' local volatility models of Derman and Kani (1994) and Rubinstein (1994). Finally hybrid models that mix local volatility with stochastic volatility or jump features (e.g. Carr *et al.* 2004) are typically not scale-invariant because of the local volatility component. The SABR model of Hagan *et al.* (2002) is also not scale-invariant.

3. A classification of hedging properties

Merton (1973) observed that the prices of vanilla calls and puts are homogeneous of degree one in the underlying price S and exercise price K when the price process is scale-invariant. More recently Bates (2005) showed that if the price of a European or American claim is homogeneous of degree one in S and K, then the option delta and gamma are model-free and uniquely determined by:

$$\delta = \frac{\partial f}{\partial S} = \frac{1}{S}\left(f - K\frac{\partial f}{\partial K}\right),$$
$$\gamma = \frac{\partial^2 f}{\partial S^2} = \left(\frac{K}{S}\right)^2 \frac{\partial^2 f}{\partial K^2}, \qquad (1)$$

where f is the *observable* option price and the partial derivatives with respect to K can be approximated via finite differences or interpolation.

Alexander and Nogueira (2007) generalized Bates' result and proved that the price hedge ratios of *any* contingent claim with a homogeneous pay-off of degree k are model-free in the class of scale-invariant models, and†

$$\delta = \frac{\partial f}{\partial S} = \frac{1}{S}\left(kf - K\frac{\partial f}{\partial K}\right),$$
$$\gamma = \frac{\partial^2 f}{\partial S^2} = \left(\frac{K}{S}\right)^2 \frac{\partial^2 f}{\partial K^2} + \frac{k-1}{S^2}\left(kf - 2K\frac{\partial f}{\partial K}\right), \qquad (2)$$

Thus if the market prices of a claim are observable and a scale-invariant price process is appropriate, all price hedge ratios—delta, gamma and higher order—of any homogeneous pay-off are model-free. They can be expressed in terms of sensitivities that can be readily obtained from the market data by interpolation or some other smoothing technique. As a result, no model is necessary and no calibration is necessary, which is a major simplification when there is no analytical solution for the claim price! We only need to use a scale-invariant price process, which anyway should always be the case for traded assets, and have a claim whose pay-off at expiry is homogeneous.‡

Besides, static replication allows model-free hedge ratios for actively traded options to be extended to model-free hedge ratios for over-the-counter (OTC) instruments whose prices are not directly observable. Due to the linearity of the derivative operator, the OTC claim hedge ratios are just a linear combination of the hedge ratios of vanilla calls and puts with the same weights as in the replicating portfolio (Derman *et al.* 1995, Carr *et al.* 1998, Allen and Padovani 2002). Hence model-free price sensitivities apply to virtually all claims on traded assets.

Why then do we often observe differences between the deltas (and gammas) from two different models? If both models are scale invariant the *only* reason their price hedge ratios differ is because they have a different fit to market prices of options. Notably, the Black–Scholes (BS) model is scale-invariant but it cannot fit all vanilla option prices with a single volatility parameter. Consequently the BS hedge ratios are quite different from those of smile-consistent models.

Bates (2005) showed that the difference between the BS delta and the delta from a smile-consistent model depends on the slope of the implied volatility smile. In equity markets where the smile typically has a pronounced negative skew, the BS delta for a vanilla option is less than the smile-consistent delta except perhaps for high-strike options where the smile may slope upwards. More generally, the delta and gamma of a standard European option in a scale-invariant smile-consistent model are given by

$$\delta = \delta^{\mathrm{BS}} - \upsilon^{\mathrm{BS}} \frac{K}{S}\frac{\partial \theta}{\partial K},$$
$$\gamma = \gamma^{\mathrm{BS}} - \left(\frac{K}{S}\right)^2 \left(\upsilon^{\mathrm{BS}} \frac{\partial^2 \theta}{\partial K^2} + 2\frac{\partial \upsilon^{\mathrm{BS}}}{\partial K}\frac{\partial \theta}{\partial K} + \kappa^{\mathrm{BS}}\left(\frac{\partial \theta}{\partial K}\right)^2\right), \qquad (3)$$

where δ^{BS}, γ^{BS}, υ^{BS} and κ^{BS} are the Black–Scholes delta, gamma, vega and kappa, respectively, and θ is the implied volatility of the option.

†For an extension of (2) that allows for other option characteristics in the same dimension as the underlying price, such as barriers or other exercise prices, refer to Alexander and Nogueira (2006). A proof of model-free higher-order price sensitivities is also provided there.

‡In practice, most claims have pay-offs that are homogeneous of degree one. This includes standard forward contracts, vanilla options, cash-or-nothing, look-backs and look-forwards, forward-starts, barriers, Asians and compound options. But binary options and power options have pay-offs that are homogeneous of degrees other than one.

The empirical results in Alexander and Nogueira (2007) show that the BS delta and gamma perform better for hedging equity index options than the model-free hedge ratios from smile-consistent models. Moreover, there is evidence that the BS delta over-hedges equity index options (see also Coleman *et al.* 2001), but the model-free delta over-hedges even more than the BS delta.

4. Minimum variance hedging

The minimum variance delta is the amount of the underlying asset that reduces to zero the instantaneous covariance of a delta-hedged portfolio with the underlying asset price (Bakshi *et al.* 1997, Lee 2001). It accounts for the total effect of a change in the underlying price, including the indirect effect of the price change on the volatility or any other parameter that is correlated with the underlying price.

In many stochastic volatility models the minimum variance hedge ratios differ from the hedge ratios obtained from the usual partial derivatives. The exception is when price-volatility correlation is zero, such as in the models of Hull and White (1987) and Stein and Stein (1991). In these models, the minimum variance delta and gamma are equal to the standard (model-free) hedge ratios given by (2). This is true whenever no other variable is correlated with the underlying price.

In equity options there is strong evidence of negative price-volatility correlation even after accounting for jumps in the price process (Nandi 1998, Andersen *et al.* 2002). Therefore, a model such as that of Heston (1993), which allows for negative correlation, should provide a better fit to equity option prices and more accurate minimum variance hedge ratios than stochastic volatility models with zero correlation. The minimum variance hedge ratios for the Heston model are derived in Alexander and Nogueira (2007).

The 'stochastic-$\alpha\beta\varrho$' model of Hagan *et al.* (2002) has recently become popular amongst practitioners. The model takes the CEV functional form (Cox 1975) for the dynamics of the forward price F and allows the alpha parameter to be driven by a correlated diffusion as follows:

$$\begin{aligned} dF &= \alpha F^{\beta} dW, \\ d\alpha &= \upsilon\alpha dZ, \quad \langle dW, dZ\rangle = \varrho dt. \end{aligned} \tag{4}$$

The model is not scale-invariant unless $\beta = 1$. The minimum variance delta and gamma (with respect to F) of a claim whose price is $g = g^{\mathrm{sabr}}(t, F; \mathbf{K}, T, \alpha, \beta, \upsilon, \varrho)$ are given by

$$\delta_{mv}^{\mathrm{sabr}} = \frac{dg}{dF} = \frac{\langle dg, dF\rangle}{\langle dF, dF\rangle} = g_F + g_\alpha \frac{\varrho\upsilon}{F^{\beta}} \tag{5}$$

$$\gamma_{mv}^{\mathrm{sabr}} = \frac{d^2 g}{dF^2} = g_{FF} + \frac{\varrho\upsilon}{F^{\beta}}\left(\left(\frac{\varrho\upsilon}{F^{\beta}}\right) g_{\alpha\alpha} + 2g_{F\alpha} - \frac{\beta}{F} g_\alpha\right). \tag{6}$$

On the right-hand side, g_F and g_{FF} are the standard delta and the standard gamma of the SABR model with respect to F. They are not model-free because the model is not scale-invariant. The second term in (5) captures the correlation between F and α, as in other stochastic volatility models.

Figure 2 compares the deltas of a selection of models obtained for the June 2004 European call options on the S&P 500 index. The date, 21 May 2004, was chosen as a day when all the hedge ratios exhibited their typical pattern, and all models were calibrated to the observed implied volatilities on that date. We consider both standard and minimum variance (MV) deltas and gammas of the following models.

(i) The Black–Scholes (BS) model is a scale-invariant non-smile-consistent model with constant volatility. Since all parameters are constant, both standard and MV hedge ratios are the same.
(ii) The Heston model is a scale-invariant model with non-zero price-volatility correlation. Its standard hedge ratios are the same as the model-free scale-invariant ratios computed from (1) or (3), but the MV hedge ratios are different.
(iii) The SABR model is a non-scale-invariant model with non-zero price-volatility correlation. Neither the MV hedge ratios nor the standard ratios are model-free since (1) does not hold.
(iv) The CEV model is a non-scale-invariant local volatility model. Since the local volatility is static, both standard and MV deltas are the same.

The call deltas of each model, whilst changing daily, exhibit some strong patterns when they are plotted by strike or by moneyness. The model-free delta, labelled 'SI' for scale-invariant, is greater than the Black–Scholes delta for all but the very high strikes. The SABR model delta lies between the Black–Scholes and model-free (SI) deltas. So if the Black–Scholes model over-hedges in the presence of the skew, which appears to be the case, then both model-free and SABR deltas will perform worse than the BS model.

But a different picture emerges when minimum variance hedge ratios are used. In both the CEV and SABR models (which are not scale-invariant) and the Heston model (which is scale-invariant) the minimum variance deltas are generally lower than the Black–Scholes deltas.

A pattern is also observed in figure 3, which compares the standard and the minimum variance gammas of the same four models. The model-free (SI) and SABR gammas are lower than the Black–Scholes gamma for in-the-money calls and greater than the Black–Scholes gamma for out-of-the-money calls (except for very deep out-of-the-money calls) while the opposite is observed when minimum variance gammas are considered.

Bakshi *et al.* (1997) find that the Black–Scholes model hedges vanilla options reasonably well if a vega (or gamma) hedge is also put in place. Alexander and Nogueira (2007) find that a delta–gamma hedge strategy based on the minimum variance ratios for vanilla options perform at least as well as the BS model. The apparent superiority of the Black–Scholes model for delta–gamma

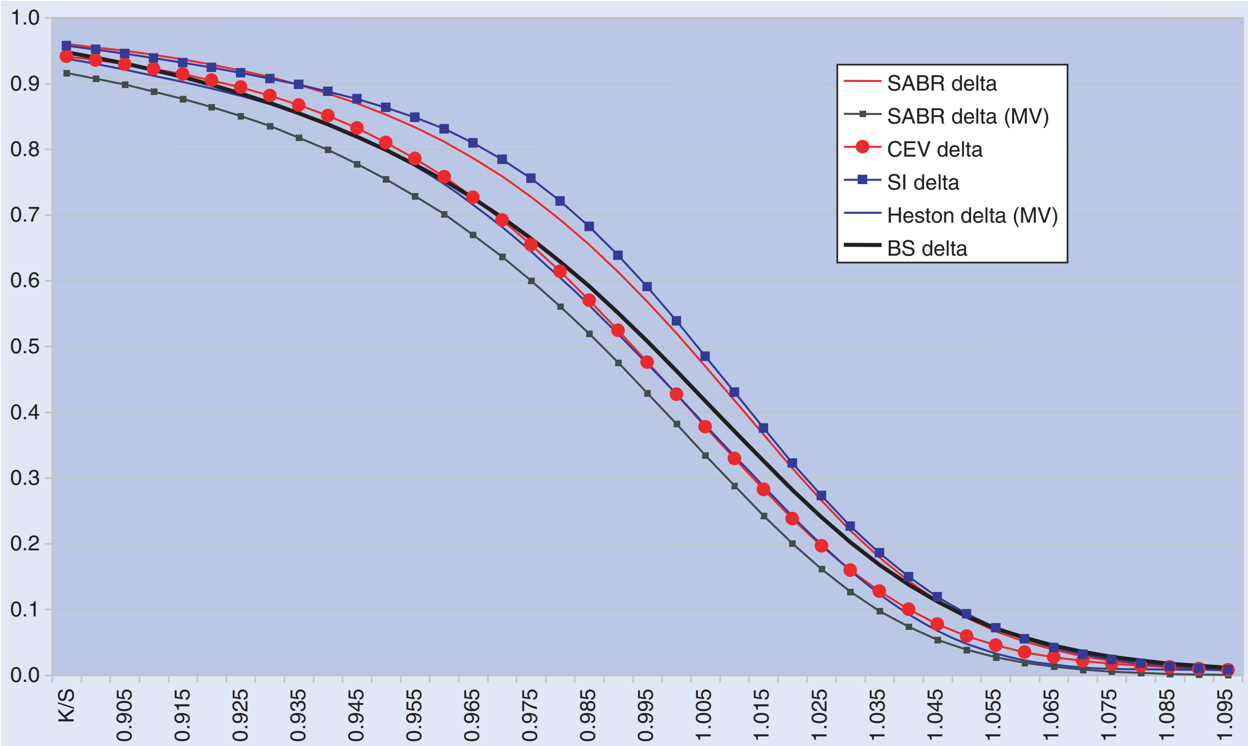

Figure 2. The models' delta by moneyness on 21 May 2004.

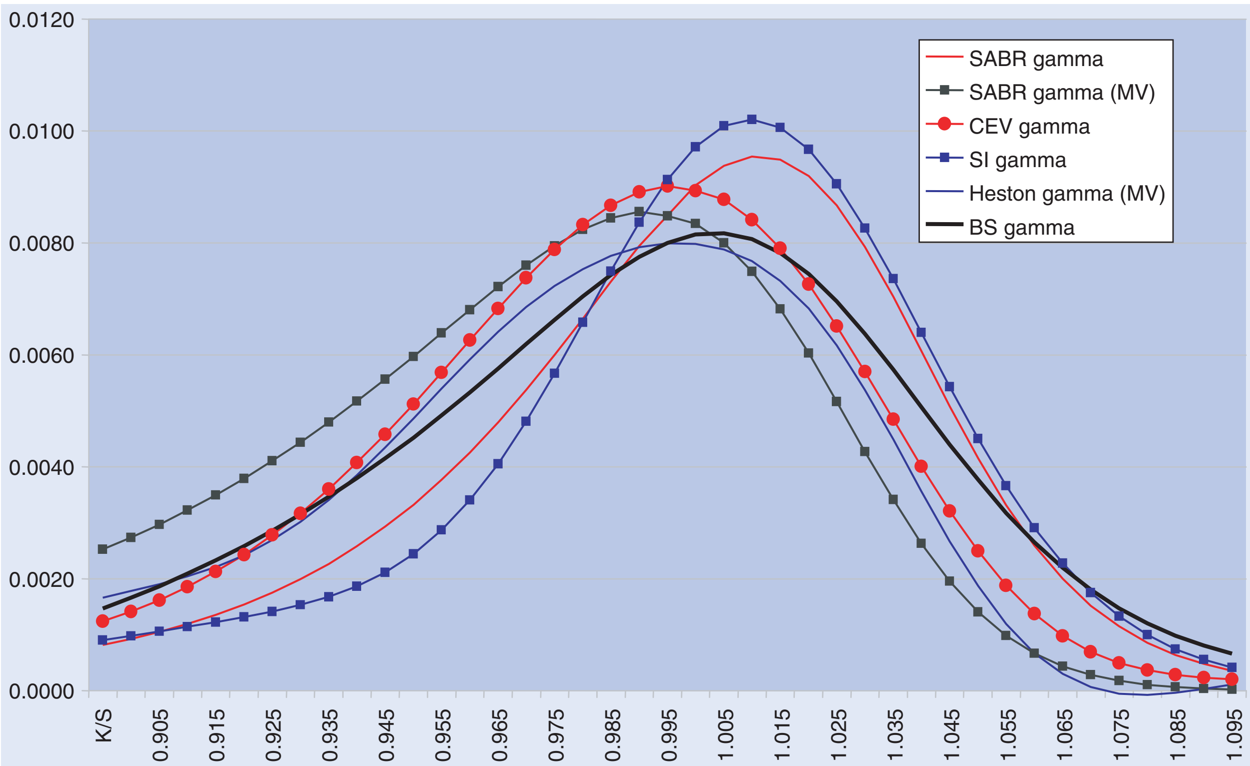

Figure 3. The models' gamma by moneyness on 21 May 2004.

hedging is due to its success at hedging the strikes close to at-the-money. For ITM and OTM vanilla calls, the minimum variance hedge ratios tend to perform best. However, overall there is no significant difference between the hedging performances of the minimum variance hedge ratios considered.

The question arises whether minimum variance deltas and gammas might also be model-free at least within the class of scale-invariant models. Using asymptotic methods Lee (2001) derives an approximation to the minimum variance delta of a stochastic volatility model that is surprisingly similar to the model-free delta in (3), except that the negative sign in the last term is replaced by a positive sign. It remains an interesting question whether Lee's model-free approximation could be generalized to other scale-invariant models.

5. Summary and conclusions

Recent research has shown that most pricing models produce identical price hedge ratios for virtually any option on tradable assets. Any difference that is apparent when these models are calibrated is only due to fitting error. And the few models that do not have theoretically identical hedge ratios may be unsuitable for modelling tradable assets. This is very good news. It means that whenever the market price of a homogeneous claim is observable, either directly or via static replication, its delta, gamma and higher-order price sensitivities are model-free. They can be expressed in terms of quantities that are easily computed from the market data and no specific model is necessary.

The bad news concerns the hedging performance of the option pricing models in common use today. For delta and delta–gamma hedging equity index options we now know that every scale-invariant model will perform worse than the Black–Scholes (BS) model! All is not lost, however. Whenever there are jumps in the price process, or the volatility (or any other parameter of the model) is correlated with the underlying asset price, these model-free hedge ratios are not variance minimizing hedge ratios. Now, for hedging equity index options, minimum variance delta hedging can be much more effective than delta hedging with the BS model. More work needs to be done in this area, but delta–gamma hedging with minimum variance hedge ratios may even provide a better performance than the BS model.

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