Operational Risk Aggregation

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Loss model approaches are currently a focus of operational risk software development, for estimating both regulatory and economic operational risk capital. The simplest of these is the "Internal Measurement Approach" (IMA) – so called by the Basel Committee (2001a, b) – where the operational risk capital requirement for a particular business line and risk type is a multiple "gamma" of the expected loss in this category. Alexander (2003) provides industry wide "gamma" factors for each risk type and line of business, using their straightforward dependence on the expected loss frequency "lambda".

Table 1: Calibration of Gamma

Lamda	**100**	**50**	**40**	**30**	**20**	**10**
99.9%-ile	131.805	72.751	60.452	47.812	34.714	20.662
phi	**3.180**	**3.218**	**3.234**	**3.252**	**3.290**	**3.372**
gamma	**0.318**	**0.455**	**0.511**	**0.594**	**0.736**	**1.066**
Lamda	**8**	**6**	**5**	**4**	**3**	**2**
99.9%-ile	17.630	14.449	12.771	10.956	9.127	7.113
phi	**3.405**	**3.449**	**3.475**	**3.478**	**3.537**	**3.615**
gamma	**1.204**	**1.408**	**1.554**	**1.739**	**2.042**	**2.556**
Lamda	**1**	**0.9**	**0.8**	**0.7**	**0.6**	**0.5**
99.9%-ile*	4.868	4.551	4.234	3.914	3.584	3.255
phi	**3.868**	**3.848**	**3.839**	**3.841**	**3.853**	**3.896**
gamma	**3.868**	**4.056**	**4.292**	**4.591**	**4.974**	**5.510**
Lamda	**0.4**	**0.3**	**0.2**	**0.1**	**0.05**	**0.01**
99.9%-ile*	2.908	2.490	2.072	1.421	1.065	0.904
phi	**3.965**	**3.998**	**4.187**	**4.176**	**4.541**	**8.940**
gamma	**6.269**	**7.300**	**9.362**	**13.205**	**20.306**	**89.401**

* Source: Operational Risk: Regulation, Analysis and Management, edited by C. Alexander (March, 2003). Table reproduced by kind permission of Pearson Education (Financial Times-Prentice Hall). For lamda less than 1, interpolation over both lamda and x has been used to smooth the percentiles; even so small non-monotonicities arising from the discrete nature of percentiles can remain.

Table 1 also shows a parameter termed "phi" which is the ratio of the percentile "unexpected loss" to the standard deviation of the loss distribution. Phi varies much less than gamma does as the loss frequency changes. There is a simple relationship between gamma, lambda and phi. In fact gamma = phi/√lambda.

Pezier and Pezier (2001) show how the IMA can be extended to include loss severity uncertainty. The IMA formula can also be generalized in a number of other ways, for example to assume different functional forms for the frequency distribution, and/or to allow for insurance cover (Alexander, 2002). Consequently, Alexander (2003) uses the gamma tables, with a severity uncertainty adjustment, and different assumptions about the loss frequency density, to show that the risk capital estimate when calculated under the "generalized IMA" formula is identical to the risk capital estimate when estimated using the simulation based "Loss Distribution Approach" (LDA) when log severity is normally distributed. Any difference is due to simulation error, and the "generalized IMA" analytic formula is more exact.[1]

So what is the point in using the LDA? The LDA has one very important advantage. In the LDA the whole loss distribution is simulated, for each risk type and line of business, and this allows the use of aggregation methods that are more appropriate than the aggregation methods that are admissible with the IMA.

The IMA gives only an estimate of the "unexpected loss", that is, the difference between an upper percentile of the loss distribution and the expected loss, which can be translated to a standard deviation using an assumed value for phi. Standard deviations can be aggregated under assumptions about the correlations between different loss distributions. For example, assuming perfect correlation between all risk types and all lines of business implies the aggregation is a simple sum of all the standard deviations. Alternatively, assuming zero correlations implies the standard deviation of the total loss is the square root of the sum of the individual variances. In between these two extremes one might attempt to specify a correlation matrix **C** that represents the correlations between different operational risks. This is an heroic assumption, about which we shall say more later. Nevertheless, suppose the $(n + m) \times (n + m)$ correlation matrix **C** is given. We have the $(n + m) \times (n + m)$ diagonal matrix **D** of standard deviations σ_{ij}, that is **D** =

[1] The inclusion of loss severity variability always increases the risk capital estimate for any given risk type and line of business, by a factor of $\sqrt{(1 + (\sigma/\mu)^2)}$ where σ is the (log) severity standard deviation and μ is the (log) severity mean. Since loss severity *is* very uncertain, particularly for high impact rare events, so σ/μ is large and the LDA risk capital estimate – which includes loss severity uncertainty – will easily be double the estimate under the basic IMA formula.

diag(σ_{11}, σ_{12}, σ_{13}, …….σ_{21}, σ_{22}, σ_{23}, …, σ_{nm},) and the $n + m$ vector φ of “phi” multipliers. Now the total unexpected loss, accounting for correlations, is Sqrt (φ '**DCD** φ)

Dependencies between Operational Risks:

Correlation is not necessarily a good measure of the dependence between two random variables. Correlation only captures linear dependence, and even in liquid financial markets, correlations can be very unstable over time. They are intrinsically a short-term measure, because they are based on short memory processes, such as financial returns or P&Ls. A huge amount of model risk is introduced by compounding risks that are assessed under a correlation measure to a long term horizon, such as the one-year horizon that many banks use for their economic capital assessments. Therefore for all risk types, and for operational risks in particular, is it more meaningful to consider general co-dependencies of loss distributions, rather than to restrict the relationships between losses to simple correlation measures.

How should a bank specify the dependence structure between different operational risks? The dependencies between operational risks may be linked to the likely movements in common attributes, that is, to the common risk drivers of these operational losses. Examples of key risk drivers are volume of transactions processed, product complexity, and staffing (decision) variables such as pay, training, recruitment and so forth. Knowing the management policies that are targeted for the next year, a bank should identify the likely changes in key risk drivers resulting from these management decisions. In this way the probable dependence structures across different risk types and lines of business can be identified.

For example, if a bank were to rationalize the back office with many people being made redundant, this would affect risk drivers such as transactions volume, staff levels, skill levels, and so forth. The consequent difficulties with terminations, employee relations and possible discriminatory actions would increase the “Employment Practices & Workplace Safety” risk. The reduction in personnel in the back office could lead to an increased risk of Internal and External Fraud, since fewer checks would be made on transactions, and there may be more errors in “Execution, Delivery & Process Management.” The other risk types are likely to be unaffected.

The Aggregation Algorithm:

Now suppose two operational risks are thought to be positively dependent because the same risk drivers tend to increase both of these risks and the same risk drivers tend to decrease both of these risks. In that case the two loss distributions are aggregated to a total loss distribution via a copula with positive dependency. More generally, copulas can be chosen to reflect positive or negative dependencies, that may be different in the tails than they are in the center of the distributions.

Before defining some copulas, and showing how they are used for aggregation, let us define the two-step algorithm:

(a) Find the joint density $h(x,y)$ given the marginal densities $f(x)$ and $g(y)$ and a given dependency structure. If X and Y were independent then $h(x,y) = f(x)\, g(y)$. When they are not independent, and their dependency is captured by a copula with probability density function $c(x,y)$, then the joint density function is $h(x,y) = f(x)\, g(y)c(x,y)$.

(b) Derive the distribution of the sum X + Y from the joint density $h(x,y)$. Let $Z = X + Y$. Then the probability density of Z is the 'convolution integral'

$$k(z) = \int_x h(x, z-x)dx = \int_y h(z-y, y)dy$$

The algorithm can be applied to find the sum of any number of random variables: if we denote by X_{ij} the random variable that is the annual loss of the line of business (i) and risk type (j), the total annual loss has the density of the random variable $X = \sum_{i,j} X_{ij}$. The distribution of X is obtained by first using steps (a) and (b) of the algorithm to obtain the distribution of $X_{11} + X_{12} = Y_1$, say, then these steps are repeated to obtain the distribution of $Y_1 + X_{13} = Y_2$ say, and so on.

Choosing the Copula to Reflect the Type of Dependency:

An approximation to the joint density if two random variables is $h(x,y) = f(x)\, g(y)c(J_1(x), J_2(y))$. The standard normal variables J_1 and J_2 are defined by $J_1(x) = \Phi^{-1}(F(x))$ and $J_2(x) = \Phi^{-1}(G(x))$ where Φ is the standard normal distribution function, F and G are the distributions functions of X and Y and

$$c\left(J_1(x), J_2(y)\right) = \frac{1}{\sqrt{1-\varrho^2}} \exp\left\{-\frac{\left[J_1(x)^2 + J_2(y)^2 - 2\varrho J_1(x) J_2(y)\right]}{2\left(1-\varrho^2\right)}\right\} \exp\left\{-\frac{\left[J_1(x)^2 + J_2(y)^2\right]}{2}\right\}$$

This is the density of the Gaussian copula. It can capture positive, negative or zero correlation between X and Y. In the case of zero correlation $c(J_1(x), J_2(y)) = 1$ for all x and y. Note that annual losses do not need to be normally distributed for us to aggregate them using the Gaussian copula. However, a limitation of the Gaussian copula is that dependence is determined by correlation and is therefore symmetric. In particular the Gaussian copula underestimates the tail dependencies that are likely to arise with operational losses.

The Gumbel copula is useful for capturing asymmetric tail dependence, for example, where there is a greater dependence between large losses than there is between small losses. It can be parameterized in two ways. Write $u = F(x)$ and $v = G(y)$, then the Gumbel δ copula density is:

$$\exp(-((-\ln u)^{\delta} + (-\ln v)^{\delta})^{1/\delta})(((-\ln u)^{\delta} + (-\ln v)^{\delta})^{1/\delta} + \delta - 1)\left(\ln u \ln v\right)^{\delta-1}\left(uv\right)^{-1}\ \ ((-\ln u)^{\delta} + (-\ln v)^{\delta})^{(1/\delta)-2}$$

In the Gumbel δ copula there is increasing positive dependence as δ increases and less dependence as δ decreases towards 1 (the case $\delta = 1$ corresponds to independence).

For the Gumbel α copula the density is given by:

$$\exp\left(-\alpha\left(\ln u\ \ \ln v / \ln\left(uv\right)\right)\right)\left((1-\alpha\left(\ln u / \ln\left(uv\right)\right)^{2})(1-\alpha\left(\ln v / \ln\left(uv\right)\right)^{2}) - 2\alpha \ln u \ln v\ \ / \left(\ln\left(uv\right)\right)^{3}\right)$$

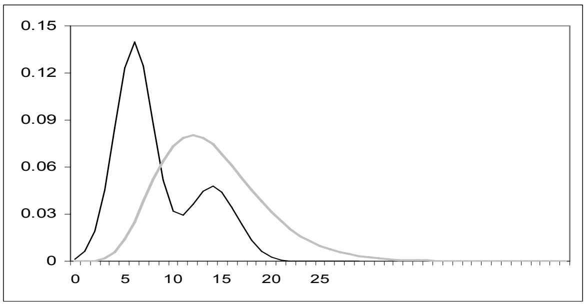
In the Gumbel α copula there is increasing positive dependence as α increases and less dependence as α decreases towards 0 (the case $\alpha = 0$ corresponds to independence).

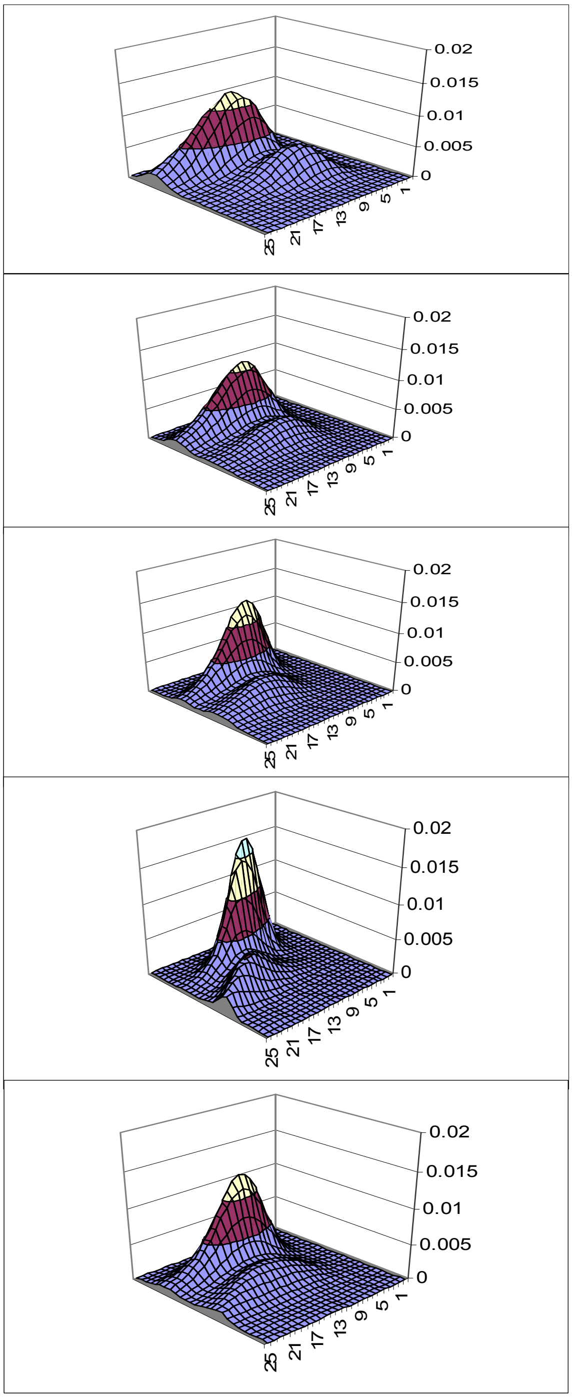
Many other copulas have been formulated, some of which have many parameters to capture more than one type of dependence. For example, a copula may have one parameter to model the dependency in the tails, and another to model dependency in the center of the distributions. More details may be found in Bouyé et. al. (2000), Frachot et. al. (2001) and Nelsen (1998).

<u>Example: Aggregating Two Operational Losses:</u>

The following example illustrates the how the type of dependency that is assumed affects the total risk. Consider the two annual loss distributions with density functions shown in figure 1. Using these for X and Y, joint densities have been obtained using the Gaussian copula with $\varrho = 0.5, 0, -0.5$ respectively; the Gumbel δ copula with $\delta = 2$ and the Gumbel α copula with $\alpha = 0.5$.

Figure 1: Annual Losses and Joint Densities





The bi-model density has been fitted by a mixture of two normal densities: with probability 0.3 the normal has mean 14 and standard deviation 2.5 and with probability 0.7 the normal has mean 6 and standard deviation 2. The other annual loss is gamma distributed with $\alpha = 7$ and $\beta = 2$.

NB figures (a) to (e) on right correspond to:

- (a) $\rho = -.05$
- (b) $\rho = 0$
- (c) $\rho = 0.5$
- (d) $\delta = 2$
- (e) $\alpha = 0.5$

Figures 2 and 3 illustrate step (b) of the aggregation algorithm, when convolution is used on the joint densities to obtain the density of the sum of the two random variables. Figure 2 shows the density of the sum in each of the three cases for the Gaussian copula, according as $\varrho = 0.5, 0, -0.5$ and figure 3 shows the density of the sum under the Gumbel copulas, for $\delta = 2$ and $\alpha = 0.5$ respectively.

Figure 2: The total loss distribution under different assumptions for correlation

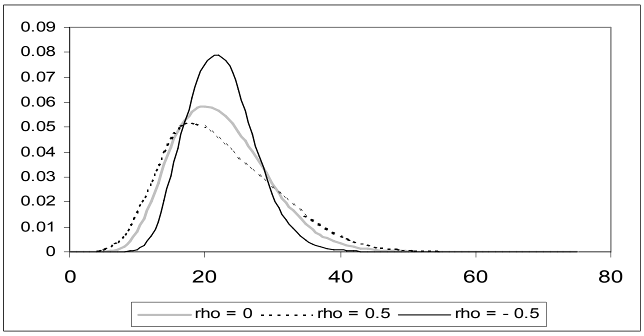


Figure 3: The total loss distribution under different assumptions about the tail dependency

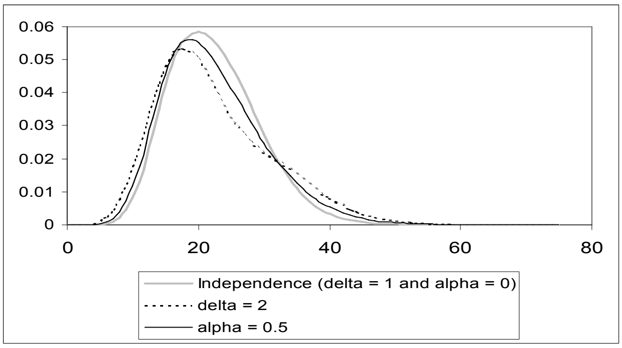


Table 2 shows that the expected loss is hardly affected by the assumptions made about co-dependencies of these two risks: it is approximately 22.4 in each case. However the unexpected loss at the 99.9^{th} percentile (and at the 99^{th} percentile) is very much affected by the assumption one makes about dependency.

Table 2: Risk Capital Estimates based on the Same Two Losses under Different Dependency Assumptions

	$\rho = -0.5$	$\rho = 0$	$\rho = 0.5$	$\delta = 2$	$\alpha = 0.5$
Expected Loss	22.3909	22.3951	22.3977	22.3959	22.3977
99.9^{th} Percentile	41.7658	48.7665	54.1660	54.9715	57.6023
Unexpected Loss	19.3749	26.3714	31.7683	32.5755	35.2046

The values of the dependence parameters were chosen arbitrarily in this example. Nevertheless, it has shown that small changes in the dependency assumption can produce estimates of unexpected total loss that is doubled or halved, even when aggregating only two annual loss distributions. Obviously the effect of dependency assumptions on the aggregation of many annual loss distributions to the total annual loss for the firm will be quite enormous.

Summary and Conclusion:

The LDA is unnecessary for estimating unexpected losses within one given risk type and line of business. The result should be similar to the result obtained using a generalized IMA formula that includes loss severity variability and an appropriate assumption about the form of loss frequency density. In fact, if log severity is assumed to be normally distributed, any differences would be due to simulation errors and it is the generalized IMA formula that is the precise analytic solution. Therefore, if large differences are observed between the LDA and the generalized IMA estimates for unexpected loss, an obvious reason for this would be that the gamma factors have not been correctly calibrated. A table for gamma factors (without loss severity uncertainty) is given in this article.

Readers that are familiar with the usual loss model framework (see Klugman, Panjer and Willmot, 1998) will understand that the IMA and the LDA are *not* two different approaches. The IMA is just an analytic formula for the unexpected loss in the compound distribution, and the LDA is just a computational method for compounding frequency and severity densities. So why use simulation? The reason lies in the aggregation of operational loss distributions to obtain the total risk capital requirement – economic and/or regulatory – for the bank. For this we need the entire compound loss distribution for each risk type and line of business in the aggregation – not just an analytic formula for the unexpected loss in the distribution. And it is better to find the compound distribution by simulation, than to attempt to infer it from the unexpected loss estimate under assumptions about moments and the functional form.

This article has described an aggregation methodology that takes account of the dependencies between operational losses arising when there are common risk drivers associated with the two losses. We have given a simple example to show that enormous differences between estimates of total economic or regulatory capital may arise, depending on the nature of these dependencies.

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