

# ARE NASH BARGAINING WAGE AGREEMENTS UNIQUE? AN INVESTIGATION INTO BARGAINING SETS FOR FIRM-UNION NEGOTIATIONS

By C. O. ALEXANDER and W. LEDERMANN

*School of Mathematical Sciences, University of Sussex, Falmer, Brighton  
BN1 9QN*

The shapes of the bargaining sets for firm-union negotiations are rigorously analyzed in two cases, according to whether bargaining is over wages alone (with employment set according to the labour demand schedule) or over both wages and employment. When bargaining concerns wages only the bargaining set becomes a one-dimensional curve, and so the standard theory of the Nash bargaining solution does not apply. Therefore existence and uniqueness conditions are proved directly. The Kalai-Smorodinsky solution is also analysed, and results are illustrated using a constant elasticity example.

## 1. Introduction

THE NASH bargaining solution (NBS) has long been used to model wage negotiations between firms and unions—examples include de Menil (1971), McDonald and Solow (1981), Nickell and Andrews (1983), Anderson and Devereux (1989), Dowrick (1989, 1990), Leslie (1990), and Hoel (1990). Much of this research has chosen quite arbitrary values for the parameters of the NBS—the *status-quo* point and the bargaining power index. However, the completion of the Nash programme by Binmore *et al.* (1986) has helped more recent research to interpret these parameters correctly within the context of wage bargaining.

But the correct identification of the NBS parameters is not the only mathematical problem that has been glossed over in economic applications of Nash bargaining theory. Generally, existence of a unique Nash bargaining solution is implicitly assumed in the literature, whether bargaining be in two-dimensions (over both wages and employment) or over wages alone with the firm reserving the right-to-manage, setting employment unilaterally according to its labour demand schedule. In this paper we show that, although concavity of the union's utility function guarantees uniqueness in the two-dimensional problem, sufficient conditions for existence of a unique solution to the constrained bargaining problem are much more complex.

In particular, we could impose strict monotonicity conditions upon the wage elasticity of the unions' utility function, the wage elasticity of employment, and the labour elasticity of the firm's revenue function to ensure uniqueness of the NBS when negotiations concern wages alone. Alternative sufficient conditions are also discussed and our results are illustrated with an example in which the wage elasticity of employment and the unions' risk aversion coefficient are both constant.

During the course of this work we have developed a general method for investigating the shape of the bargaining set for any type of revenue and utility function. It turns out that this method is very useful in the analysis of any bargaining solution (such as the Kalai-Smorodinsky) to negotiations which may concern both wages and employment levels, or only wages. The method is explained in the appendix and used in the constant elasticity example of section three.

## 2. Non-uniqueness in the right-to-manage model

We consider the usual sort of utility function for models of firm-union bargaining. The union has utility  $V(\omega, L)$  which depends on both employment  $L \in \mathbb{R}$  and wages  $\omega \in \mathbb{R}$ . Specifically, following McDonald and Solow (1981), and Dowrick (1990), we set

$$V(\omega, L) = Lu(\omega)$$

where  $u$  is a twice differentiable utility function,  $u'(\omega) > 0$  and  $u''(\omega) < 0$ . The unions' utility is normalized so that  $u(\omega^*) = 0$  for the competitive (or reservation) wage  $\omega^*$ . The firm's utility value is its profit, *viz.* the excess of revenue over costs. So

$$\Pi(\omega, L) = R(L) - \omega L$$

where  $R$  is twice continuously differentiable,  $R(0) = 0$ ,  $R'(L) > 0$  and  $R''(L) < 0$ .

Now suppose that a firm and union negotiate only the wage, and that firms reserve the right to manage by setting employment unilaterally. Profit maximization yields the labour demand schedule  $R'(L) = \omega$  which will act as a constraint on the wage negotiations and the firm's aspiration level is

$$\Pi^* = \Pi(\omega^*, L^*)$$

where  $L^*$  is such that  $R'(L^*) = \omega^*$ . In fact the bargaining set becomes the image of the labour demand curve, which is a compact curve  $\Gamma$  in  $(\Pi, V)$  space (see Fig. 1).

Before investigating conditions under which no unique NBS to this problem exists, we must take a much closer look at this bargaining curve. First, we show that natural assumptions about the wage elasticity of  $V$  imply only that the curve is downwards sloping along its Pareto optimal boundary, and not that it is convex: for unions to engage in wage bargaining it is natural to assume that at some points along the labour demand curve, an increase in wages will not decrease the total utility of the union  $V$ , even though employment will decrease. Equivalently, the wage elasticity of  $V$ ,  $\delta(\omega)$ , is positive. However, this assumption only guarantees that  $d\Pi/dV$  will be negative along  $\Gamma$ .

*Proposition 1* Along  $\Gamma$ ,  $d\Pi/dV < 0$  whenever  $\delta(\omega) > 0$ .

*Proof* We may parametrize  $\Gamma$  by  $L$ , as  $\Pi(L) = R(L) - R'(L)L$  and

$V(L) = Lu(R'(L))$  where  $0 \leq L \leq L^*$ . Now  $d\Pi/dL = -LR''(L)$  and  $dV/dL = u(R'(L)) + Lu'(R'(L))R''(L)$ . Hence  $d\Pi/dV = -LR''/(u + u'LR'')$  so

$$\frac{d\Pi}{dV} < 0 \iff \frac{u'\omega}{u} > \frac{-R'}{LR''}$$

Denote by  $\eta(\omega)$  the wage elasticity of employment, namely

$$\eta(\omega) = -d\ln L/d\ln \omega = -R'/LR''$$

Now let  $\varepsilon(\omega) = d\ln u/d\ln \omega$ , so that

$$\delta(\omega) = \frac{\partial \ln V}{\partial \ln \omega} = \frac{\partial \ln u}{\partial \ln \omega} + \frac{\partial \ln L}{\partial \ln \omega} = \varepsilon(\omega) - \eta(\omega)$$

and the condition reduces to  $\delta(\omega) > 0$ . □

Of course it is not realistic to suppose that the elasticity of  $V$  with respect to  $\omega$  is constant. Typically it will decrease as the wage increases along the labour demand curve and may become negative for high values of  $\omega$ . It is not unreasonable to assume that  $\delta(\omega)$  is positive for small  $\omega$ , and then decreases monotonically as we move up the labour demand curve. Now as the wage increases along the labour demand curve so, by Proposition 1, we move down  $\Gamma$  until the point where  $\delta(\omega) = 0$ . If no such  $\omega$  exists then  $\Gamma$  never bends back on itself. But if there exists a finite  $\omega$  such that  $\delta(\omega) = 0$  then at this point  $\varepsilon = \eta$ , and the union achieves its aspiration utility level  $\hat{V}$ . Hereafter  $\Gamma$  bends back on itself, as  $\delta(\omega)$  becomes negative. As the wage increases further the continuity assumptions

$$\lim LR'(L) = \lim Lu(R'(L)) = 0 \tag{1}$$

as  $L \rightarrow 0$ , guarantee that  $\Gamma$  ends at the origin, as shown in Fig. 1(b). These assumptions are not obvious when  $R'(L) \rightarrow \infty$  as  $L \rightarrow 0$ , and they are not necessary if we restrict the parameter  $L$  to the half-open interval,  $(0, L^*]$ . However, we prefer to assume that  $0 \leq L \leq L^*$  and make the assumptions (1), which will also guarantee continuity of the Nash objective function at the origin.

We now investigate conditions for a unique Nash solution to this one-dimensional bargaining problem: the existence theory of Nash (1950) holds only for two-dimensional bargaining sets. Denote by  $\hat{L}$  the value of  $L$  for which  $\Gamma$  bends back on itself, and set  $\hat{L} = 0$  if no such positive  $\hat{L}$  exists. Now the problem is to find  $L$  to maximize the Nash objective function  $G(L) = V(L)^\beta \Pi(L)$ , that is

$$G(L) = [Lu(R'(L))]^\beta [R(L) - LR'(L)] \tag{2}$$

*Theorem 1* A unique Nash solution  $L_N$  exists in  $[\hat{L}, L^*]$  if

- (i)  $G''(L) < 0$ ; or
- (ii)  $d^2\Pi/dV^2 < 0$  when  $\delta(\omega) > 0$ ; or
- (iii)  $\varepsilon(\omega)$  is monotonic decreasing in  $\omega$ ,  $\eta(\omega)$  is monotonic increasing in  $\omega$ , and  $d\ln R/d\ln L$  is monotonic decreasing in  $L$ .

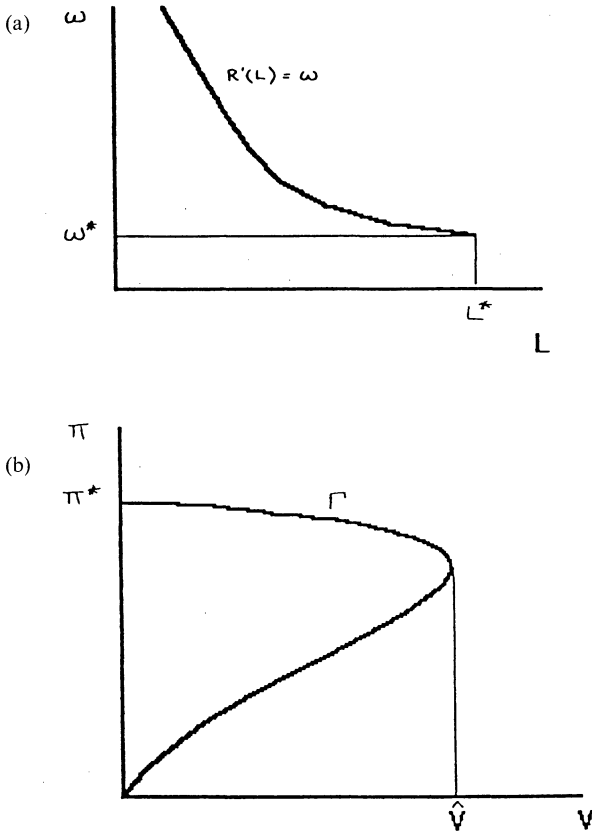


FIG. 1. (a) The labour demand schedule; (b) The bargaining curve

*Proof* (i) Since  $\Pi = 0$  when  $L = 0$  and  $d\Pi/dL = -R''L > 0$  along the labour demand curve, the assumptions (1) imply that  $G(L) \geq 0$  in  $[0, L^*]$ . Now if  $G''(L) < 0$  then  $G'(L)$  is a monotonic decreasing function and hence cannot vanish more than once. If  $G'(L) = 0$  for an interior point  $L_N \in (\hat{L}, L^*)$  then the Nash solution may be found as the unique solution to

$$\text{Max}_L V(\omega, L)^\beta \Pi(\omega, L)$$

such that

$$R'(L) = \omega$$

The first order condition yields

$$u(\omega_N) = \frac{\beta}{\zeta} \frac{\hat{v}}{\Pi^*} \left( \frac{R(L_N)}{L_N} - R'(L_N) \right)$$

where  $\zeta$  is the absolute value of slope of the common tangent between the bargaining curve and the Nash objective. Thus the workers' utility at the Nash solution is proportional to the difference between the average and marginal products of labour. If  $G'(L)$  does not vanish and hence does not change sign, then  $L_N = L^*$  if  $G'(L) > 0$  and  $L_N = \hat{L}$  if  $G'(L) < 0$ . In all cases a unique value  $L_N$  in  $[\hat{L}, L^*]$  exists.

(ii) The proof is given in Alexander and Ledermann (1994).

(iii) To prove this we first take logarithmic derivatives of (2), which yields

$$G'/G = \beta/L + \beta(u'/u)R'' - LR''/(R - LR')$$

Since  $G(L_N) \neq 0$  and  $R'' \neq 0$  in  $[0, L^*]$  the first order condition is

$$\beta/LR'' + \beta(u'/u) - L/(R - LR') = 0$$

or, since  $R' > 0$

$$\beta(R'/LR'') + \beta(R'u'/u) - LR'/(R - LR') = 0 \quad (3)$$

If we assume that  $\varepsilon(\omega)$  is a monotonic decreasing function of  $\omega$  then  $\varepsilon$  is a monotonic increasing function of  $L$ , that is  $R'u'/u \uparrow$  (as a function of  $L$ ). If we assume that  $\eta(\omega)$  is a monotonic increasing function of  $\omega$  then it is a monotonic decreasing function of  $L$ , that is  $-R'/LR'' \downarrow$ , so  $R'/LR' \uparrow$ . To guarantee that the last term of (3) is also a monotonic increasing function of  $L$ , we assume that  $R'L/R \downarrow$ . For

$$-LR'/(R - LR') \uparrow \Leftrightarrow (R - LR')/LR' \uparrow$$

since the average product is always greater than the marginal product of labour. But the right hand term is monotonic increasing  $\Leftrightarrow R/LR' \uparrow \Leftrightarrow LR'/R \downarrow$ . Hence the three monotonicity assumptions imply monotonicity of the left hand side of (3), and hence exactly one solution.  $\square$

Of the three alternative sufficient conditions in Theorem 1, the first is the least restrictive, but lacks both economic and geometric intuition. The second condition is that the Pareto optimal segment of  $\Gamma$  be convex upwards, and is similar to the standard convexity assumptions of the original two-dimensional problem (Nash, 1950). The conditions (iii) are much stronger than (i), but have more economic intuition. They show that if reasonable monotonicity assumptions are made about the wage elasticity of employment, the wage elasticity of individual's utility, and the labour elasticity of the firm's revenue function, then existence of a unique constrained Nash solution is guaranteed. The assumption of monotonicity of the wage elasticity of employment does, in fact, impose a positive upper bound on  $R'''$ , viz.

$$R''' \leq (R'')^2/R' - R''/L$$

as can be seen by differentiating  $\eta$  with respect to  $L$ . Similarly a negative upper bound on  $R''$  is imposed by the monotonicity assumption for  $d \ln R / d \ln L$ , viz.  $R'' \leq (-R'/RL)(R - R'L)$ .

### 3. A constant elasticity example

In this section we use the method outlined in the appendices to take a detailed look at the shape of the bargaining set for both unconstrained and constrained bargaining problems when specific functional forms are assigned to the firm's revenue function and union's utility function. We then show that the conditions for a unique NBS are satisfied by this choice of functions, whether bargaining be over both wages and employment or over wages alone. We assume that

$$u(\omega) = \frac{(\omega - \omega^*)^a}{a}$$

where  $0 < a < 1$ , so that risk aversion in the union is given by the constant  $a$ . We also assume that

$$R(L) = L^b/b$$

where  $0 < b < 1$ , so that the wage elasticity of employment is also constant

$$\eta(\omega) = -R'/LR'' = (1 - b)^{-1}$$

In this case, with the notation defined in Appendix 1, we have

$$L^* = \omega^{*1/(b-1)}, \bar{\omega} = \omega^*/(1 - a) \quad \text{and} \quad \varepsilon(\omega) = a\omega/(\omega - \omega^*)$$

(thus  $\varepsilon(\omega) \downarrow 1$  as  $\omega \rightarrow \bar{\omega}$ ). The equation of the contract curve is

$$aL^{b-1} = (a - 1)\omega + \omega^*$$

and since now  $\Pi = L^b/b - \omega L$  and  $V = L(\omega - \omega^*)^a/a$  we have, along the contract curve, that

$$\Pi = (1 - a + ab)L^b/b(1 - a) - \omega^*L/(1 - a)$$

and

$$V = (a/(1 - a))^a(\omega^* - L^{b-1})^aL/a$$

Hence

$$d\Pi/dV = -((1 - a)/a)^{a-1}(\omega^* - L^{b-1})^{1-a}$$

Thus, at  $\Pi^*$ ,  $d\Pi/dV = 0$ , otherwise  $d\Pi/dV < 0$ . Now

$$d/dL(d\Pi/dV) = -(a/(1 - a))^{1-a}(1 - a)(\omega^* - L^{b-1})^{-a}(1 - b)L^{b-2} < 0$$

and

$$dV/dL = (a/(1 - a))^a(\omega^* - L^{b-1})^{a-1}(\omega^* - (1 + ab - a)L^{b-1})/a > 0$$

Hence

$$d^2\Pi/dV^2 = d/dL(d\Pi/dV)/dV/dL < 0$$

which verifies that a unique Nash solution to the firm-union negotiation problem over both wages and employment exists (see Appendix 1).

Now consider the bargaining curve when negotiations concern wages only, with employment set by the firm according to the labour demand schedule.

In this case

$$\delta(\omega) = \frac{a\omega}{(\omega - \omega^*)} - (1 - b)^{-1}$$

hence  $\delta(\omega)$  is infinitely positive at  $\omega^*$ , monotonically decreasing thereafter and  $\delta(\omega) = 0$  when  $\omega$  is given by  $\omega_m = \omega^*/(1 - a + ab)$ . So  $\Gamma$  is horizontal at  $\omega^*$ , and then  $d\Pi/dV < 0$  along  $\Gamma$  as  $\omega$  increases up to  $\omega_m$ ;  $\Gamma$  is vertical at  $\omega = \omega_m$  and then bends back on itself as  $d\Pi/dV$  becomes positive. In this case, along the labour demand curve

$$\Pi = ((1 - b)/b)L^b$$

and

$$V = (\omega - \omega^*)^a L/a$$

So the firm and the unions' aspiration levels of utility are  $\Pi^* = \omega^{*-d}/d$  where  $d = b/(1 - b)$  and  $\hat{V} = \omega^{*-e} a^{a-1} (1 - b)^a (1 - a + ab)^e$  where  $e = (1 - a + ab)/(1 - b)$ . Hence the constraint curve  $\Gamma$  may be parametrized by  $\omega$ , as follows

$$x = K(\omega - \omega^*)^a (\omega/\omega^*)^{1/(b-1)} \omega^{*-a}$$

and

$$y = (\omega/\omega^*)^{-d}$$

where  $K$  is a positive constant which depends only on  $a$  and  $b$ , and the coordinates  $x$  and  $y$  have been normalized (by dividing by the union's and the firm's aspiration levels respectively) so that  $0 \leq x, y \leq 1$ . Differentiating yields

$$\frac{dy}{dx} = \frac{M\omega(\omega - \omega^*)^{1-a}}{\omega(1 - a + ab) - \omega^*}$$

where  $M$  is a positive constant which depends on  $a$ ,  $b$ , and  $\omega^*$ . Thus  $\Gamma$  is horizontal at the top, when  $\omega = \omega^*$  and  $L = L^*$ , and vertical when  $\omega = \omega_m$  and  $L = L^*(1 - a + ab)^{1/(b-1)}$ . By construction, at this point  $x = 1$  and  $y = (1 - a + ab)^d$ . The shape of  $\Gamma$  in normalized coordinates is shown in Fig. 2 and the conditions of theorem 1(iii) may easily be verified for the Pareto optimal segment of  $\Gamma$ , that is for  $\omega \in [\omega^*, \omega_m]$ . This proves the existence of a unique constrained Nash solution.

In Appendix 2 the relative wage at the Nash and Kalai-Smorodinsky solutions is determined, and shown to be independent of the reservation wage. To illustrate this result, consider the case  $a = 1/2$ ,  $b = 1/3$ . Now the normalized coordinates are given by

$$x = (\sqrt{27}/2)\omega^*\omega^{-3/2}(\omega - \omega^*)^{1/2}$$

and,

$$y = (\omega^*/\omega)^{1/2}$$

Hence

$$\frac{dy}{dx} = \frac{2\omega(\omega - \omega^*)^{1/2}}{\sqrt{27}\omega^{*1/2}(2\omega - 3\omega^*)}$$

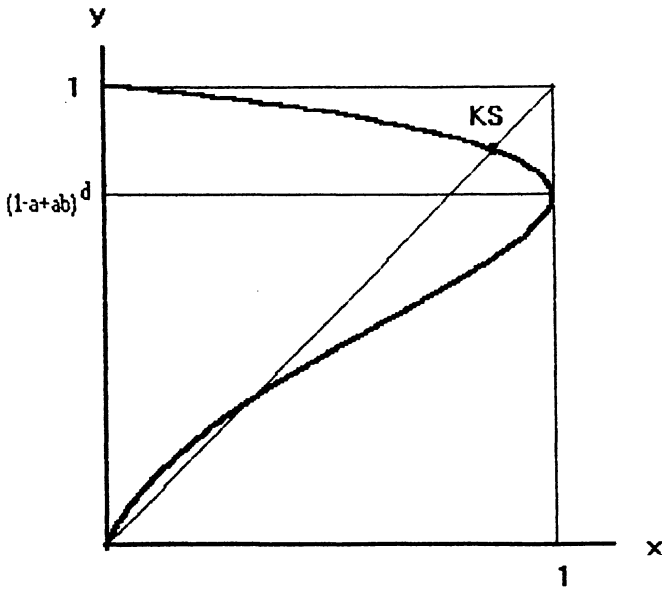


FIG. 2. The bargaining curve in normalized coordinates

and it is easy to show from this that  $d^2y/dx^2 < 0$  on the Pareto optimal section of  $\Gamma$ , which is therefore strictly convex upwards. Now

$$\omega_N = \frac{\omega^*(1 + 3\beta)}{(1 + 2\beta)}$$

and

$$\omega_{ks} = \omega^*(27 - \sqrt{297})/8$$

and

$$\left(\frac{dy}{dx}\right)_N = -2(1 + 3\beta)[\beta/27(1 + 2\beta)]^{1/2}$$

so by the result of Appendix 2

$$\omega_N \cong \omega_{ks} \Leftrightarrow \beta \cong 0.39538$$

APPENDIX 1

1. Determining the shape of a bargaining set

The bargaining set in  $(\Pi, V)$  space is given by

$$B = \{V(\omega, L), \Pi(\omega, L) | (\omega, L) \in F\}$$

where  $F$  is the set of feasible wage-employment pairs. Figure 3 illustrates the mapping  $\phi$  between  $(\omega, L)$  space and the bargaining set in  $(\Pi, V)$  space. If negotiations do not reach agreement, so that no union members are employed by the firm and the firm makes zero profit, the *status quo* point  $(\omega^*, 0)$  is mapped to the origin of the bargaining set.

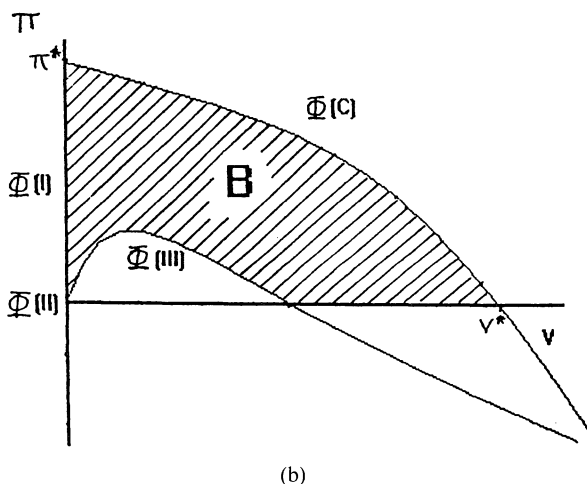
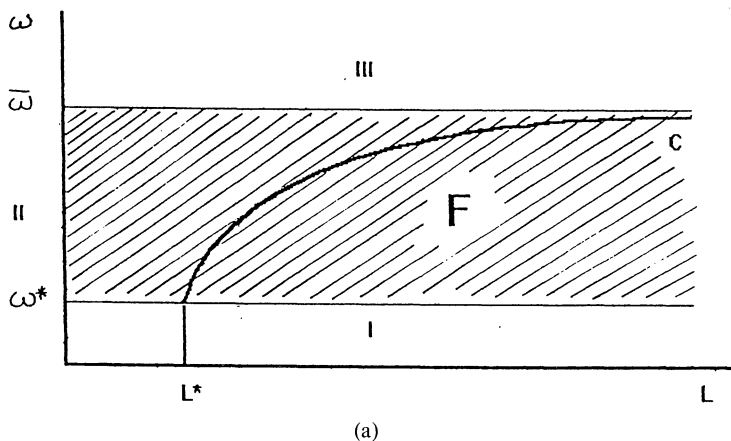


FIG. 3. (a) The feasible set; (b) The bargaining set

McDonald and Solow (1981) define the contract curve in  $(\omega, L)$  as the locus of points of tangency between the profit and union indifference curves; thus

$$\Pi_L / \Pi_\omega = V_L / V_\omega$$

and so the Jacobian,  $J$ , of the transformation  $\{\Pi(\omega, L), V(\omega, L)\}$  is zero along the contract curve. Now, by definition, the contract curve is mapped to the efficient boundary of the bargaining set, so the inverse function theorem tells us that the transformation has a local inverse (so that points in the bargaining set have a unique image in  $(\omega, L)$  space) except along the efficient boundary. Since  $J = (-L u')(R' - \omega + u/u')$  the equation of the contract curve is

$$R'(L) = \omega - u(\omega)/u'(\omega) = \omega(1 - 1/\varepsilon(\omega)) \tag{4}$$

where  $\varepsilon$  denotes the wage elasticity of  $u$ . Thus wages are a mark-up on the marginal product of labour, since the union is a monopoly supplier of labour. Since  $R'(L) > 0$  we have that  $\varepsilon(\omega) > 1$

along the contract curve. If we assume further that  $R'(L) \rightarrow 0$  as  $L \rightarrow \infty$  then also  $\varepsilon(\omega)$  converges to unity as we move up the contract curve. If there exists a point  $\bar{\omega}$  such that  $\varepsilon(\bar{\omega}) = 1$ , the curve will become asymptotic to the line  $\omega = \bar{\omega}$ . This gives rise to a feasible set

$$F = \{(\omega, L) | \omega^* \leq \omega \leq \bar{\omega}, 0 \leq L \leq \infty\} \tag{5}$$

which contains the contract curve. Since  $dL/d\omega = uu''/R''u' > 0$  along the contract curve it is indeed monotonic increasing as shown in Fig. 3(a).

Of course if no finite  $\bar{\omega}$  such that  $\varepsilon(\bar{\omega}) = 1$  exists, then the feasible space has no upper bound for  $\omega$ . But before illustrating the feasible set in this case we consider the shape of the bargaining set  $B$  for a feasible set such as (5). Boundaries of  $F$  will be mapped to boundaries of  $B$  under the transformation  $\{\Pi(\omega, L), V(\omega, L)\}$ , so consider in turn the boundaries (I), (II) and (III) marked on Fig. 3 and their images  $\Phi(I)$ ,  $\Phi(II)$  and  $\Phi(III)$  in  $(\Pi, V)$  space.

(i)  $\omega = \omega^*, 0 \leq L < \infty$

When  $\omega = \omega^*$  then  $V = 0$  so  $\Phi(I)$  lies on the  $\Pi$ -axis. It begins at the origin when  $L = 0$  and, assuming  $R'(0) > \omega^*$ , it increases to a maximum given by

$$\Pi^* = \text{Max} ((R(L) - \omega^*L))$$

Thus at  $\Pi^*$ ,  $L$  is such that the marginal product of labour equals the competitive wage, that is  $L = L^*$  and  $R'(L^*) = \omega^*$ . Thereafter  $\Phi(I)$  moves down the  $\Pi$ -axis, becoming negative as  $L \rightarrow \infty$ .

(ii)  $\omega^* \leq \omega \leq \bar{\omega}, L = 0$

When  $L = 0$  so also  $\Pi = V = 0$  and so  $\Phi(II)$  lies at the origin (status-quo) of the bargaining set.

(iii)  $\omega = \bar{\omega}, 0 \leq L < \infty$

When  $\omega = \bar{\omega}$  then  $\Pi = R(L) - \bar{\omega}L$  and  $V = Lu(\bar{\omega})$  so  $d\Pi/dV = (R'(L) - \bar{\omega})/u(\bar{\omega})$  and  $d^2\Pi/dV^2 = R''(L)/u(\bar{\omega})^2 < 0$ . The segment  $\Phi(III)$  is therefore 'convex upwards' and has one turning point, when  $L$  is such that  $R'(L) = \bar{\omega}$ . If  $R'(0) > \bar{\omega}$  then  $d\Pi/dV > 0$  for small  $L$  and so  $\Phi(III)$  will cross the  $V$  axis. This must occur when  $L$  is such that the average product of labour equals the maximum wage. Now as  $L \rightarrow \infty$ ,  $V$  becomes infinitely positive and  $\Pi$  infinitely negative along  $\Phi(III)$ .

1.1. (C) the contract curve

The Pareto optimal boundary of the bargaining set is given by the image curve,  $\Phi(C)$ ; it is not possible for  $\Phi(III)$  to cross  $\Phi(C)$  because the wage is always greater along III than along C for a given level of employment. When  $\omega = \omega^*$ , at the beginning of the contract curve, then also  $L = L^*$ ,  $\Pi = \Pi^*$  and  $V = 0$ . Thus the contract curve has an image which intersects the  $\Pi$ -axis at the firm's aspiration point (see Fig. 3(b)). It is easy to show that  $d\Pi/dV < 0$  along  $\Phi(C)$ , for  $d\Pi/dL = -u/u' - Ld\omega/dL$ ,  $dV/dL = u + Lu'd\omega/dL$  and since  $d\omega/dL = R''u^2/uu'' > 0$  we have  $d\Pi/dL < 0$ ,  $dV/dL > 0$ , and so  $d\Pi/dV < 0$ . Moreover, since  $d\Pi/dV = -1/u'$

$$d/dL(d\Pi/dV) = u''(\omega)d\omega/dL/u'^2 = R''/u < 0$$

and so  $d^2\Pi/dV^2 < 0$ , and the Pareto optimal boundary is convex. Now suppose that  $\varepsilon(\omega) \downarrow 1$  as  $\omega \rightarrow \infty$ , so that the feasible set has no upper bound for  $\omega$ . Provided still that  $R'(0) > \omega^*$  (which is a perfectly reasonable assumption) the segments  $\Phi(I)$  and  $\Phi(C)$  of the bargaining set are the same as in Fig. 3(b). But  $d\Pi/dV < 0$  along  $\Phi(III)$ , which therefore remains below the  $V$ -axis, and the bargaining set will be all points in the positive quadrant bounded by  $\Phi(I)$  and  $\Phi(C)$ .

When this type of technique is used to determine the shape of a bargaining set, existence of a unique solution for any particular utility can be proved (or disproved) directly. For example, it is straightforward to show that concavity of  $u(\omega)$  is sufficient to guarantee existence of a unique NBS (a result which is stated but not proved in MacDonald and Solow, 1981).

**Appendix 2**

**1. The Constrained Kalai-Smorodinsky solution and its relation to the Constrained Nash solution**

Since the existence proof of Kalai and Smorodinsky (1975) does not apply to one-dimensional bargaining we prove existence directly: the normalized coordinates of  $\Gamma$  are

$$x(L) = \frac{V(L)}{V(\hat{L})}, \quad y(L) = \frac{\Pi(L)}{\Pi(L^*)}$$

where  $V(\hat{L}) = \max V(L)$ . The Kalai-Smorodinsky (KS) solution is given by  $L_{ks} \in [\hat{L}, L^*]$  satisfying  $K(L_{ks}) = 0$ , where  $K(L) = x(L) - y(L)$ . Note that  $K(\hat{L}) > 0$  and  $K(L^*) < 0$  so there exists at least one such  $L_{ks}$ . In order to prove uniqueness consider the derivative

$$K'(L) = V'(L)/V(\hat{L}) - \Pi'(L)/\Pi(L^*)$$

Since  $\delta(\omega) > 0 \Rightarrow V'(L) < 0$ , and  $\Pi'(L) > 0$  for all  $L \in [\hat{L}, L^*]$ ,  $K'(L) < 0$ . Hence  $K$  is monotonic decreasing and cannot have more than one zero in  $[\hat{L}, L^*]$ . It follows that  $L_{ks}$  is unique. At the KS solution  $\omega_{ks} = R'(L_{ks})$  and

$$u(\omega_{ks}) = \frac{\hat{V}}{\Pi^*} \left( \frac{R(L_{ks})}{L_{ks}} - R'(L_{ks}) \right)$$

so the worker's utility will again be proportional to the difference between the average and marginal products of labour, just as it is for the Nash solution. Indeed, the wage at the Nash solution is greater than that at the KS solution if and only if the gradient of the bargaining curve at the Nash solution is less than the index of bargaining power. More precisely:

*Proposition 2* Denote by  $\beta$  the unions' index of bargaining power, so that the Nash objective function is

$$N(\omega) = x^\beta y, \beta > 0,$$

and let

$$\left( \frac{dy}{dx} \right)_N$$

denote the gradient of  $\Gamma$  at the point  $\omega = \omega_N$ . Then

$$\omega_{ks} \cong \omega_N \Leftrightarrow \left| \left( \frac{dy}{dx} \right)_N \right| \cong \beta$$

*Proof* Since  $N'(\omega_N) = 0$  and

$$\left( \frac{dy}{dx} \right)_N = \frac{y'(\omega_N)}{x'(\omega_N)} \text{ we have } y(\omega_N) = -\frac{1}{\beta} \left( \frac{dy}{dx} \right)_N x(\omega_N).$$

Let  $K(\omega) = x(\omega) - y(\omega)$ . Then

$$K(\omega_N) = x(\omega_N) - y(\omega_N) = x(\omega_N) \left\{ 1 + \frac{1}{\beta} \left( \frac{dy}{dx} \right)_N \right\}$$

However,  $K'(\omega) = x'(\omega) - y'(\omega) < 0$  and so  $K'(\omega) \cong 0 \Leftrightarrow \omega \cong \omega_{ks}$ . Thus

$$\omega_{ks} \cong \omega_N \Leftrightarrow \frac{1}{\beta} \left( \frac{dy}{dx} \right)_N \cong -1.$$

Since

$$\left(\frac{dy}{dx}\right)_N < 0$$

we have proved the result.  $\square$

Thus, the relative magnitude of  $\omega_N$  and  $\omega_{ks}$  depends on a constant which is a function of the parameters of the utility function. More generally, the higher is the unions' index of bargaining power, the more likely that it will prefer the Nash solution and this is independent of its reservation wage.

#### REFERENCES

- ALEXANDER, C. O. and LEDERMANN, W. (1994). 'The constrained Nash bargaining solution', *Journal of the Operational Research Society*, **45**, 954–8.
- ANDERSON, S. P. and DEVEREUX, M. (1989). 'Profit sharing and optimal labour contracts', *Canadian Journal of Economics*, **22**, 425–33.
- BINMORE, K., RUBINSTEIN, A. and WOLINSKY, A. (1986). 'The Nash bargaining solution in economic modelling', *Rand Journal of Economics*, **17**, 176–88.
- DOWRICK, S. (1989). 'Union-oligopoly bargaining', *Economic Journal*, **99**, 1123–42.
- DOWRICK, S. (1990). 'The relative profitability of Nash bargaining on the labour demand curve or the contract curve', *Economics Letters*, **33**, 121–5.
- HOEL, M. (1990). 'Local versus central wage bargaining with endogenous investments', *Scandinavian Journal of Economics*, **92**, 453–69.
- KALAI, E. and SMORODINSKY, M. (1975). 'Other solutions to Nash's bargaining problem', *Econometrica*, **43**, 513–8.
- LESLIE, D. (1990). 'A disequilibrium model under bilateral monopoly', *Bulletin of Economic Research*, **42**, 155–74.
- MCDONALD, I. M. and SOLOW, R. M. (1981). 'Wage bargaining and employment', *American Economic Review*, December, 896–908.
- DE MENIL, G. (1971). 'Bargaining: monopoly power versus union power', MIT Press, Cambridge, MA.
- NASH, J. F. (1950). 'The bargaining problem', *Econometrica*, **18**, 155–62.
- NICKELL, S. J. and ANDREWS, M. (1983). 'Unions, real wages and employment in Britain, 1951–79', *Oxford Economic Papers*, **35**, 183–206.