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Modelling Regime-Specific Stock Price Volatility*

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Abstract

Single-state generalized autoregressive conditional heteroscedasticity (GARCH) models identify only one mechanism governing the response of volatility to market shocks, and the conditional higher moments are constant, unless modelled explicitly. So they neither capture state-dependent behaviour of volatility nor explain why the equity index skew persists into long-dated options. Markov switching (MS) GARCH models specify several volatility states with endogenous conditional skewness and kurtosis; of these the simplest to estimate is normal mixture (NM) GARCH, which has constant state probabilities. We introduce a state-dependent leverage effect to NM-GARCH and thereby explain the observed characteristics of equity index returns and implied volatility skews, without resorting to time-varying volatility risk premia. An empirical study on European equity indices identifies two-state asymmetric NM-GARCH as the best fit of the 15 models considered. During stable markets volatility behaviour is broadly similar across all indices, but the crash probability and the behaviour of returns and volatility during a crash depends on the index. The volatility mean-reversion and leverage effects during crash markets are quite different from those in the stable regime.

I. Introduction

Modelling regimes in market volatility is of interest to risk managers in banks, asset managers and other financial firms. For the assessment of risk capital and for setting

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traders' limits, risk managers need to understand how volatility may respond differently to market shocks during a predominantly stable period, compared with a crash or recovery period. When computing economic capital, and to enable traders' limits to be raised when markets are considered to be exceptionally volatile, risk models are subjected to 'stress tests'. These tests are also required by most banking regulators, and will be more closely tied to regulatory risk capital under the new Basel accord. However, they are often performed using *ad hoc* adjustments in the absence of a precise modelling framework. The inadequacy of banks' risk capital during the financial crisis in 2008 has highlighted the urgent need to improve stress testing methodology. The tests should be able to specify an appropriate intensity for shocks to returns and volatility, to model the response to these shocks and to quantify the probability associated with a market crash scenario. This paper presents a regime-dependent volatility model which encompasses all these effects, and whose parameters are straightforwardly estimated using historical data on stock market returns.

Financial risk managers are not the only audience for this paper: governments, regulators and traders in the capital markets may also be interested. Modelling volatility following large price falls in an index is essential for governments and policy-makers as well, because they need great skill to manage the economy in volatile times. At the time of writing, the governments were fire-fighting a systemic risk in which insolvency in the banking sector had spilled over into other sectors of the economy, as the worst financial crisis since the great depression heightens the possibility of a prolonged global recession. Because the over-riding priority for blanking regulators is to avert mass insolvency, they need to understand how markets behave during crashes. Finally, an understanding of the best fitting discrete time volatility model should help traders to select from the plethora of possible stochastic volatility models available, many of which will fit the market-implied volatility smile equally well.

A main characteristic of volatility is 'mean-reversion', whereby volatility persists after a market shock but in the absence of any further shocks it will eventually return to a long-run volatility level. Mean-reversion is captured in the standard discrete-time approach to modelling volatility, that is, the generalized autoregressive conditional heteroscedasticity (GARCH) framework introduced by Engle (1982) and Bollerslev (1986).[1] Many authors, such as Bekaert and Wu (2000), Wu (2001), and Hansen, Lunde and Nason (2003), argue that it is also important for a volatility model to capture the 'leverage effect' in stock markets, whereby volatility responds more to negative shocks than to positive shocks of the same magnitude.

[1]However, normal GARCH models do not explain why both conditional and unconditional returns on financial assets have such high skewness and kurtosis. For this reason, Bollerslev (1987) extended the GARCH model to include the Student's t-conditional distribution and Fernandez and Steel (1998) enhanced it further using the skewed t-distribution.

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Following Nelson (1991), a large number of alternative GARCH processes with leverage effects are available.[2] However, the vast majority of the models that have been proposed are single-state volatility models, which assume that the GARCH process has regime-invariant parameters. Hence, they do not allow different types of shocks to precipitate different types of responses. Yet a rumour may have an enormous effect but die out very quickly, whereas an announcement of important changes to economic policy (if represented by a shock of the same size as the rumour) may produce a less volatile but more persistent response, after which the general level of volatility is raised. In a single-state GARCH process, the mean-reversion and leverage effects cannot be state-dependent so, if multiple states exist, the estimated parameters can represent only an average of these effects.

Bates (1991), Christoffersen, Heston and Jacobs (2006) and others argue that, for a model to capture the empirical characteristics of option implied volatility skews, it is essential to account for time variability in the physical conditional skewness and kurtosis. However, single-state GARCH models have constant conditional skewness and conditional kurtosis, so the option implied volatility smiles and skews that are generated from these models are unrealistic. In contrast, when there is more than one volatility state in a GARCH model, both conditional skewness and kurtosis can be endogenously time-varying.

Multi-state GARCH models have their origins in the Markov switching models of Hamilton and Susmel (1994), Cai (1994) and Gray (1996) and numerous enhancements of this early work have recently been suggested in a burgeoning literature.[3] Perhaps the most tractable Markov switching GARCH model was introduced by Haas, Mittnik and Paolella (2004b) but even this is quite difficult to estimate.[4] Our focus is on the regime-specific asymmetric behaviour of the variance process. For

[2]These include the absolute-value GARCH based on Taylor (1986, pp. 78–79) and Schwert (1989), the asymmetric and nonlinear asymmetric GARCH introduced by Engle (1990) and Engle and Ng (1993), the GJR model of Glosten, Jagannathan and Runkle (1993), the AGARCH model of Ding, Granger and Engle (1993), the threshold GARCH of Zakoïan (1994), the quadratic GARCH (QGARCH) of Sentana (1995), the Box–Cox-transformed GARCH of Hentschel (1995) and Hwang and Basawa (2004), which nests several different models, the model of Brännäs and de Gooijer (2004) that extends QGARCH and, more recently, the model of Lanne and Saikkonen (2007) that generalizes the AGARCH model.

[3]Dueker (1997) discusses an estimation method using a collapsing procedure to approximate the likelihood function for a simplified version of MS-GARCH; Klaassen (2002) makes use of the full information set available when computing the conditional probability of a state; Marcucci (2005) allows for regime-specific means; Francq, Roussignol and Zakoïan (2001) and Francq and Zakoïan (2005, 2008) consider a model with state-dependent parameters in a single variance process; Henneke, Rachev and Fabozzi (2006) and Bauwens, Preminger and Rombouts (2006, 2007b) use Bayesian techniques to estimate this type of processes.

[4]These authors assume that each state is characterized by its own variance and that the variance process of each state depends only on its lagged values (as opposed to lagged values of the other variance processes) and the squared residuals; for this model, maximum likelihood estimation is feasible. Weak stationarity properties of the (1, 1) model and conditions for the existence of the higher moments, relaxing the assumption of initial finite variance, were discussed in Liu (2006); Abramson and Cohen (2007) also develop stationarity properties for the models of Klaassen (2002) and Haas, Mittnik and Paolella (2004b) – for models of order (p, q) – using a technique based on backward recursion. Liu (2007) introduces a general model that allows for asymmetries in the variance equations; the focus of this paper is the discussion of the stationarity properties and existence of higher moments of a generalization of the model of Haas, Mittnik and Paolella (2004b) – an MS Box–Cox-transformed threshold GARCH(1, 1), to be more precise.

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this reason we shall ignore the autocorrelation in the state variable, so that the state probabilities are not time-varying, and through this simplification we can accommodate the possibility of different means in the volatility states. We shall assume that the market shock follows a normal mixture (NM) distribution and call the model as the NM-GARCH process. It is the simplest possible framework for a multi-state GARCH model, and it is more tractable than Markov switching GARCH.[5]

The general symmetric NM-GARCH model was introduced by Haas, Mittnik and Paolella (2004a) and its properties were further investigated by Alexander and Lazar (2006). Numerous specific versions of symmetric NM-GARCH have been considered,[6] but what most of these models lack is a leverage effect in the state-dependent GARCH volatility processes. As such, they may be suitable for currency markets, but not for equity or commodity markets in which volatility responses to shocks are highly asymmetric. In symmetric NM-GARCH the only source of skewness in the physical returns densities is the different means in the normal components of the mixture conditional density. Hence such models are not suitable when the leverage effect is known to be very important, as it is in individual stock and stock index markets.

This paper examines the theoretical and empirical behaviour of state-dependent asymmetric volatility processes in an NM-GARCH model. A main strength of the model is that it allows the asymmetric volatility response to stock price shocks to be quantified in both stable and stressful market circumstances. When two-state NM models are applied to financial markets, often two states can be differentiated: a 'stable' state which occurs most of the time and a 'stressful' or 'crash' state that occurs only rarely. At every point of time the state is selected randomly, independently of the previous state. However, as we shall see later, following this random and unknown state selection and after a return is realized from the given distribution, it is possible to estimate the probability that a state was selected. This will be the *ex-post* inference about the probability of the state at time t, and it will be closely related to the actual state of the world. Also, it allows the conditional higher moments to change in time. Two asymmetric models are considered in this paper, namely the asymmetric

[5]Another reason for restricting our study to asymmetric NM-GARCH models is that MS-GARCH models do not necessarily perform better than mixture models. A thorough theoretical and empirical comparison of MS and NM-GARCH models is given by Haas, Mittnik and Paolella (2004b), by evaluating the fit of these models for three currencies. Using the BIC criterion, NM models are preferred in two out of three cases. When the forecasting performance of these models is studied, then again in most cases the NM models proved to dominate. The only criterion according to which MS models are (marginally) preferred is the fit of the conditional higher moments.

[6]Early versions of NM-GARCH models were discussed in Vlaar and Palm (1993), Ding and Granger (1996), Bauwens, Bos and van Dijk (1999) and Bai, Russell and Tiao (2001, 2003). Concentrating on the individual means in the mixture density, Wong and Li (2001) introduced an NM-ARCH model with individual AR processes in the mean equations. This model was later extended to GARCH processes by Lanne and Saikkonen (2003); they also assume that the mixing weights are functions of past observations. Bayesian estimation of the NM-GARCH was discussed in Bauwens and Rombouts (2007). Multivariate extensions were considered by Haas, Mittnik and Paolella (2006), Fong, Li and An (2006) and Bauwens, Hafner and Rombouts (2007N) – the latter uses an estimation technique based on the Expectations Maximization (EM) algorithm. Henneke *et al.* (2006) and Liu (2007) introduce leverage effects the state-dependent GARCH volatility processes.

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power GARCH (AGARCH) and GJR (Glosten, Jagannathan and Runkle) extension of NM-GARCH, based on the papers of Engle (1990), Engle and Ng (1993) and Glosten *et al.* (1993).[7] Implementing such a model should allow risk capital assessments to be more refined and help risk managers to derive regime-specific limits for traders. It should also bring new insights, relevant to investors and policy-makers, about the likelihood of a market crash and the returns/volatility behaviour during a crash period.

Our empirical study on four major European equity market indices demonstrates the superiority of a two-state asymmetric NM-GARCH model to capture the empirical characteristics of stock index returns. We also compare the risk neutral skews generated by single-component GARCH models with the skews that are generated by the two-state asymmetric GARCH models. Even without a volatility risk premium, and certainly without the need to introduce a time-varying one, the volatility smile implied by asymmetric NM-GARCH models exhibits a pronounced skew that persists into long-dated options.

The rest of the paper is organized as follows: section II defines the general asymmetric NM-GARCH model and describes the two sources of asymmetry in these models; section III characterizes the equity index data for four major equity markets and presents the estimation methodology; section IV reports our empirical results. We start with our estimations for 15 different GARCH models, including two-state asymmetric and symmetric NM-GARCH models, and symmetric and skewed t-GARCH with both symmetric and asymmetric variance processes. Then several model selection criteria are applied to identify the best model. Choosing the two-state NM-AGARCH and GJR models as best overall, the estimated parameters are used to infer various aspects of volatility behaviour in European stock markets. Section V simulates prices of European calls and puts of various maturities and compares the implied equity index skews of the FTSE100 index simulated by symmetric and asymmetric GARCH models with one and two states. Section VI summarizes and concludes.

II. The asymmetric NM-GARCH model

The model extends the GARCH processes studied in Haas *et al.* (2004a) and Alexander and Lazar (2006) to include a leverage effect in the GARCH variance equations. Specifically, the model has one equation for the mean and K variance equations. For simplicity the conditional mean equation is written as $y_t = \varepsilon_t$, but this can be easily extended to allow for explanatory variables because their coefficients can be estimated separately. The error term ε_t, which captures the market shock, is assumed to have a conditional NM density with zero mean, which is a weighted average of K normal density functions with different means and variances. We write:

[7]One of the models, the NM-GJR would be a special case of the model of Liu (2007), with $\delta = 1$ and with no autocorrelation for the state variable, except that the NM-GJR model allows for regime-specific means. The other model discussed in this paper, NM-AGARCH, is not nested within the family of models considered by Liu (2007).

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$$\varepsilon_t \mid I_{t-1} \sim NM(p_1,\ldots,p_K,\mu_1,\ldots,\mu_K,\sigma_{1t}^2,\ldots,\sigma_{Kt}^2), \quad \sum_{i=1}^{K} p_i = 1, \quad \sum_{i=1}^{K} p_i\mu_i = 0 \quad (1)$$

and the conditional density of the error term is:

$$\eta(\varepsilon_t) = \sum_{i=1}^{K} p_i \varphi_{it}(\varepsilon_t), \quad (2)$$

where φ_{it} represents normal density functions at time t with different means μ_i and different time-varying variances σ_{it}^2 for $i = 1,\ldots,K$.

The mixing law $p = (p_1,\ldots,p_K)'$ has elements that may be interpreted as the relative frequency of each state occurring over a long period of time; for reasons of identifiability we assume that $1 > p_1 \geq p_2 \geq \cdots \geq p_K \geq 0$. There are K conditional variance components and these can follow any GARCH process, but for the purpose of this paper we assume that there are three possibilities:

(i) NM-GARCH(1, 1):

$$\sigma_{it}^2 = \omega_i + \alpha_i \varepsilon_{t-1}^2 + \beta_i \sigma_{it-1}^2 \quad \text{for } i = 1,\ldots,K. \quad (3)$$

(ii) NM-AGARCH(1, 1) (based on the model of Engle, 1990 and Engle and Ng, 1993):

$$\sigma_{it}^2 = \omega_i + \alpha_i(\varepsilon_{t-1} - \lambda_i)^2 + \beta_i \sigma_{it-1}^2 \quad \text{for } i = 1,\ldots,K. \quad (4)$$

(iii) NM-GJR(1, 1) (based on the model of Glosten *et al.*, 1993):

$$\sigma_{it}^2 = \omega_i + \alpha_i \varepsilon_{t-1}^2 + \lambda_i d_{t-1}^- \varepsilon_{t-1}^2 + \beta_i \sigma_{it-1}^2 \quad \text{for } i = 1,\ldots,K, \quad (5)$$

where $d_t^- = 1$ if $\varepsilon_t < 0$, and 0 otherwise.

For the normal densities to be identified well, we also assume that:[8]

$$|\mu_i - \mu_j| + |\omega_i - \omega_j| + |\alpha_i - \alpha_j| + |\beta_i - \beta_j| + |\lambda_i - \lambda_j| > 0 \quad \text{for } i \neq j.$$

The main difference between NM-GARCH and the MS-GARCH model of Haas *et al.* (2004b) is that the latter allows for autocorrelation in the state variable; in other words, for the NM-GARCH model the transition matrix is of rank 1, so it can be expressed as:

$$P = (p_{ij}) = P[\Delta_t = i \mid \Delta_{t-1} = j] = p\mathbf{1}' \quad \text{where } 1 = (1,\ldots,1)'.$$

Also, the NM-GARCH adds asymmetry by allowing for non-zero means of the different components.

[8]Also note that we do not require that $\omega_i > 0$; actually it often happens that the ω parameter of one of the components is slightly negative. However, as long as it is small in absolute value, it does not affect the positivity of the variance series (also, see Alexander and Lazar, 2006).

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The overall conditional variance can be expressed as:

$$\sigma_t^2 = \sum_{i=1}^{K} p_i \sigma_{it}^2 + \sum_{i=1}^{K} p_i \mu_i^2 \tag{6}$$

and, given the data, an *ex-post* estimate of the probability of the ith state at an arbitrary time t, based on Bayes' rule, is given by:[9]

$$p_{i,t} = \frac{p_i \phi_i(\varepsilon_t)}{\sum_{j=1}^{K} p_j \phi_j(\varepsilon_t)}. \tag{7}$$

This is a time-varying probability, giving a realistic *ex-post* inference about the probability of state i occurring at time t. Thus at any point in time we have an *ex-ante* probability of the states, given by the mixing law, and an *ex-post* estimate of the state probabilities, given by equation (7).

The weak stationarity conditions for the two models are stated in the Appendix. When the models are restricted to NM-GARCH(1,1), then the stationarity conditions are equivalent to those given by Alexander and Lazar (2006). Also, they are equivalent to those given by Haas *et al.* (2004a) and Liu (2006) when the condition $\omega_i > 0$ is imposed (in this case the stationarity condition given in the Appendix reduces to $n > 0$ only).[10] As the NM-GJR with zero regime-specific means is a special case of the model of Liu (2007), its stationarity conditions can be deducted from Liu's results. In particular, when we assume that $\omega_i > 0$, then our stationarity condition reduces to $n > 0$, which is equivalent to the weak stationarity result of Liu (2007) restricted to NM-GJR models.[11]

There are two distinct sources of asymmetry in the model.

- *Persistent asymmetry*: This arises in all three models when the conditional returns density is a mixture of normal density components having different means; it is generated by the difference between the expected returns under different market circumstances. The Appendix shows that even the unconditional density will have non-zero skewness, and that this increases with the difference between the component means. For instance, when $K = 2$, there is positive skewness in the overall conditional returns density if the component with the higher probability has low volatility and a *negative* mean and negative skewness in the overall conditional returns density if the component with the higher probability has low volatility and a *positive* mean.[12] In our empirical results on stock indices, the negative skewness case is the one that generally predominates.

[9]For instance, consider the case $K = 2$, and suppose ε_t has a higher likelihood under φ_1 than it does under φ_2. Then the *ex-post* probability of state 1 will increase and the *ex-post* probability of state 2 will decrease.

[10]The only exception is that Liu (2006) assumes zero regime-specific means.

[11]The results of Liu (2007) apply for $\delta = 1$ and rank(P) = 1.

[12]As suggested by one of the referees, this can be seen by expressing the third moment as:

$$m_3 = p_1 p_2 [(p_2 - p_1)(\mu_1 - \mu_2)^3 + 3(\mu_1 - \mu_2)(\sigma_1^2 - \sigma_2^2)].$$

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- *Dynamic asymmetry*: This only occurs in models (ii) and (iii) and is because of the leverage parameters in the component variance processes. If the leverage parameter λ_i is positive, the conditional variance in this component is higher following a negative unexpected return than a positive unexpected return. In equity markets, where a negative unexpected return increases the leverage of a firm we expect positive values of the leverage parameter. However, some studies on single-state asymmetric GARCH (e.g. Koutmos, Negakis and Theodossiou, 1993; Chortareas, McDermott and Ritsatos, 2000) report negative leverage coefficients in stock indices of developing economies.[13]

Taken together, these two sources of skewness in the physical conditional returns density offer a much richer structure for capturing the shape of equity index skews than is given by the traditional GARCH models. The unconditional skewness and excess kurtosis are both non-zero and the conditional higher moments are also non-zero and time-varying. See the Appendix for the derivation of the unconditional higher moments.

The conditional NM densities can be interpreted as merely a device to increase the flexibility of the returns density, because the model will capture the behaviour of returns during different regimes and identify the long-run probability of each state, based on time-series data. The model is considerably easier to estimate than the class of Markov switching GARCH models introduced by Hamilton and Susmel (1994) even with the restrictions and improvements introduced by Cai (1994), Gray (1996), Klaassen (2002), and Haas *et al.* (2004b). The difficulty with estimating most of these models (except the last one) lies in the co-dependencies of the state variances. However, the NM-GARCH models considered here have a very straightforward relationship between the individual variances, because they are tied to each other only through their dependence on the error term. Hence an advantage of NM-GARCH is that quite complex volatility feedback mechanisms can be included yet the model remains easy to estimate.

III. Data and parameter estimation

Our results are based on daily closing prices of four major European equity market indices from 1 January 1991 to 21 October 2005. These are: CAC40 (total number of observations is 3,733), DAX30 (3,730), DJ Eurostoxx 50 (3,810) and FTSE100 (3,739). The sample period encompasses a number of stock market crises, including the Asian crisis in 1997, the Russian crisis in 1998, the technology market crash in 2000 and the terrorist attack on the United States in 2001.

Table 1 summarizes the general characteristics of the daily returns. The skewness is negative and the excess kurtosis is positive and in most indices these are highly significant. Note that during this period the FTSE index was less volatile in general

[13]Note that the ith variance component depends on the dispersion of the unexpected return, not around its mean μ_i in the individual density, but around the overall mean 0. Hence, there is a third effect that induces skewness in each component conditional return density but not in the overall conditional return density.

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TABLE 1

Summary statistics of the stock market indices

	CAC	*DAX*	*Eurostoxx*	*FTSE*
Volatility	21.24%	22.82%	20.13%	16.41%
Skewness	−0.099*	−0.271***	−0.166***	−0.113**
Excess kurtosis	2.86***	4.03***	4.25***	3.19***
Maximum moment exponent	5.58**	4.59***	5.66***	6.70*

Notes: The standrd errors (SE) of the sample estimates of the mean, variance, skewness and excess kurtosis parameters are as follows: SE of the sample mean $= \sigma/\sqrt{T}$, SE of the sample variance $= \sqrt{2\sigma^2/T}$, SE of the sample skewness $\approx \sqrt{6/T}$, SE of the sample excess kurtosis $\approx \sqrt{24/T}$, where T represents the total number of observations. *** represents results significantly different from zero at the 0.1% level; ** at 1% and * at 5%. The last row reports the HKKP estimate for the maximum moment exponent; here ***, ** and * denote results that are significantly higher than 4 at 0.1%, 1% and 5% significance levels, respectively.

than the other European markets. The last row of Table 1 shows a test for the existence of the fourth moment. This is based on the HKKP procedure derived by Huisman *et al.* (2001, 2002), which estimates the maximum moment exponent based on the method introduced by Hill (1975). The estimates for the maximum exponent are shown.[14] The results demonstrate that we can reject the hypothesis that the fourth moment does not exist, for all series.

For each index, we estimate the conditional variance parameters on the residuals ε_t from constant conditional mean equations.[15] Then, maximizing the likelihood function, or equivalently, maximizing

$$L(\boldsymbol{\theta} \mid \varepsilon) = \sum_{t=1}^{T} \ln[\eta(\varepsilon_t)]$$

the optimal parameter values are found, given the data. The updating formula has the following form, where $\mathbf{g}$ is the gradient vector, $\mathbf{H}$ the Hessian matrix and s represents the step-length:

$$\boldsymbol{\theta}_{m+1} = \boldsymbol{\theta}_m - s[\mathbf{H}(\boldsymbol{\theta}_m)]^{-1}\mathbf{g}(\boldsymbol{\theta}_m).$$

To compute the Hessian matrix and the gradient vector we can use either analytic or numerical first-and second-order derivatives of the likelihood – see the Appendix for further details.[16]

[14]We have used oridinary least squares for the estimation of the maximum exponent instead of weighted least squares (WLS) because the WLS procedure suggested by Huisman *et al.* (2001, 2002) actually increased the level of heteroscedasticity of the residuals. To correct for heteroscedasticity we used White's heteroscedasticity consistent covariance matrix. We used the 500 most extreme observations in the HKKP regression.

[15]We fitted autoregressive moving average (ARMA) models to the series; the best fitting model for all series was an ARMA(0, 0) model. Thus, we only removed the means of the series and continued with the demeaned series.

[16]The results were generated using C++ and Ox version 3.30 (Doornik, 2002) and the G@rch package version 3.0 (Laurent and Peters, 2002).

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IV. Empirical results

Overview of models

We fitted 3 symmetric and 12 asymmetric GARCH models to the equity index data. The first nine models have a single variance state and the last six models have two variance states.[17] The models are as follows:[18]

A. Models with normally distributed errors:
   (1) GARCH
   (2) AGARCH
   (3) GJR
B. Models with symmetric Student's t-distributed errors:
   (4) GARCH
   (5) AGARCH
   (6) GJR
C. Models with skewed Student's t-distributed errors:
   (7) GARCH
   (8) AGARCH
   (9) GJR
D. NM-GARCH models with zero means in the mixture component densities:
   (10) NM-GARCH
   (11) NM-AGARCH
   (12) NM-GJR
E. General NM-GARCH models:
   (13) NM-GARCH
   (14) NM-AGARCH
   (15) NM-GJR

The estimation results are reported in Tables 2–5.[19] The upper figure in each cell reports the parameter estimate and the lower figure is the t-ratio. Note that in these

[17]Several restricted versions of the NM-GARCH model were also fitted to the data (assuming a constant variance component or assuming constant difference between the two variance processes) – but these performed quite badly according to some of the selection criteria and thus these models are not discussed here.

[18]All models are GARCH(1, 1) specifications. To simplify notation, this will be implied in the following. Models (2), (3), (5–7) and (11–13) have one source of asymmetry, while models (8), (9), (14) and (15) have two sources. Also, all NM models will be based on two normal distributions in the mixture, as important conclusions can be drawn from such a set up – see also Marcucci (2005) and Bauwens *et al.* (2007b). We do not consider models with more than two components as those would increase the number of parameters from 10 to at least 16 and, as the results of Haas *et al.* (2004a,b) and Alexander and Lazar (2006) clearly indicate, in the case of three-component mixture models convergence is harder to achieve and also these models prove to be over-parameterized when it comes to out-of-sample forecasting.

[19]Note that the results in Tables 2–5 are for variance-annualized unexpected returns. That is, daily returns are premultiplied by $\sqrt{250}$ before estimation. Thus, volatilities are quoted in annualized terms. Note that the constant term is negative for models (5) and (8) in Table 2 and models (2), (5) and (8) in Table 5. However, the existence conditions given in Alexander and Lazar (2006) show that it is possible for an NM-GARCH model to have a finite, positive variance even when the constant term is negative. The value of the constant is very small, indistinguishable from zero, and thus the probability of getting a negative variance is effectively zero.

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TABLE 2

Estimation results for the CAC 40 index

Model	*(1)*	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*	*(13)*	*(14)*
p_1 (d.f. for				11.5408	12.2000	12.1752	12.1359	12.6697	12.7269	0.9356	0.9790	0.9765	0.9141	0.9780
(4)–(9))				(8.60)	(8.88)	(8.79)	(7.99)	(8.27))	(8.13)	(40.62)	(78.45)	(70.10)	(30.24)	(79.76)
μ_1 (skew for							−0.0724	−0.0728	−0.0751				0.0084	0.0050
(7)–(9))							(−2.96)	(−2.99)	(−3.08)				(2.48)	(2.63)
ω_1	6.4*E*–4	9.6*E*–5	5.9*E*–4	3.9*E*–4	−8.2*E*–5	4.6*E*–4	3.9*E*–4	−5.8*E*–5	4.7*E*–4	2.2*E*–4	−1.2*E*–4	4.0*E*–4	1.7*E*–4	−1.1*E*–4
	(6.43)	(0.66)	(5.84)	(3.24)	(−0.43)	(4.01)	(3.24)	(−0.31)	(4.04)	(2.44)	(−0.75)	(4.06)	(2.06)	(−0.69)
α_1	0.0663	0.0495	0.0166	0.0600	0.0545	0.0166	0.0606	0.0553	0.0173	0.0486	0.0517	0.0157	0.0446	0.0524
	(9.83)	(10.97)	(3.02)	(7.95)	(7.60)	(2.17)	(8.05)	(7.66)	(2.26)	(7.59)	(7.90)	(2.25)	(7.20)	(8.07)
λ_1		0.1060	0.0664		0.1082	0.0732		0.1067	0.0732		0.1094	0.0692		0.1058
		(7.08)	(7.91)		(5.89)	(6.25)		(5.87)	(6.26)		(6.17)	(6.57)		(6.22)
β_1	0.9186	0.9345	0.9348	0.9310	0.9319	0.9346	0.9303	0.9312	0.9340	0.9386	0.9313	0.9344	0.9432	0.9311
	(112.80)	(150.07)	(141.81)	(108.12)	(113.70)	(115.80)	(108.16)	(112.93)	(115.80)	(124.47)	(121.15)	(123.20)	(131.68)	(122.93)
p_2										0.0644	0.0210	0.0235	0.0859	0.0220
μ_2													−0.0890	−0.2235
ω_2										0.0148	0.0087	0.0174	0.0085	−0.0018
										(1.70)	(0.18)	(0.35)	(1.88)	(−0.12)
α_2										0.5600	1.6171	1.5287	0.5340	1.5497
										(0.87)	(0.52)	(0.45)	(1.33)	(0.85)
λ_2											−0.0375	−0.4967		−0.0311
											(−0.45)	(−0.33)		(−0.62)
β_2										0.6794	0.6336	0.6352	0.6893	0.6172
										(2.72)	(1.04)	(0.95)	(4.33)	(1.68)
Unconditional σ	20.55%	20.18%	19.65%	20.86%	20.22%	19.47%	20.69%	20.55%	19.79%	20.80%	21.30%	20.30%	20.81%	21.77%
Unconditional σ_1										19.46%	20.35%	19.39%	19.23%	20.74%
Unconditional σ_2										34.89%	47.97%	43.86%	31.93%	43.71%
Unconditional τ							−0.0881	−0.0861	−0.0883	−4.32*E*–6	−1.41*E*–5	−1.33*E*–5	−0.172	−0.235
Unconditional k	1.2503	0.8485	1.1207	3.3426	2.6090	4.8650	—	—	—	4.0435	12.8803	6.0332	4.8605	11.8355

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TABLE 2

(continued)

Model	*(1)*	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*	*(13)*	*(14)*
BIC	−0.4634	−0.4728	−0.4722	−0.4842	−0.4941	−0.4931	−0.4844	−0.4943	−0.4934	−0.4768	−0.4851	−0.4838	−0.4794	−0.4875
Moment tests 1%	3	3	3	2	1	1	3	1	1	4	3	3	4	1
Density	1.3101	1.4771	0.9944	0.9228	1.1713	1.5195	0.8417	0.9824	0.8340	0.7382	0.8662	0.9965	0.5757	1.1570
ACF	0.0500	0.1848	0.1033	0.3976	0.0677	0.3012	—	—	—	0.0665	0.0375	0.0463	0.1077	0.0393

Notes: Results of estimating the 15 different GARCH models using daily returns on the equity indices for the period 1 January 1991 to 21 October 2005 are reported. The total number of observations is 3,733. The first row shows the degrees of freedom in the Student t-GARCH models, and the probability of the first component in the NM-GARCH models; the second row shows the skewness in the skewed Student t-GARCH models and the mean of the first component in the general NM-GARCH models. Thereafter, we show the model's coefficient estimates and, for the NM-GARCH models, the second component's probability and mean, and the coefficient estimates of the second component. Numbers in parenthesis represent t-values.

Below this we list the unconditional volatility of each model, and for each of the components in the NM-GARCH models; the unconditional skewness and kurtosis, if they exist ['—' indicates negative kurtosis, or an unreasonably high value for the autocorrelation function (ACF) statistic, meaning that in that model the fourth moment is not defined]. Finally, we list the results of the four diagnostic tests (a)–(d) described in section IV. These are: (a) the Bayesian information criterion (BIC); (b) the moment specification tests of Newey (1985) at 1% significance; (c) the modified Kolmogorov–Smirnov statistic for the fit between the unconditional density simulated using the model and that observed empirically, in the historical data; and (d) the mean-squared error criterion to assess the fit of the model's theoretical ACF to the empirical ACF.

The models are: (1) GARCH, (2) AGARCH and (3) GJR, all three with normally distributed errors; (4) GARCH, (5) AGARCH and (6) GJR, all three with symmetric Student's t-distributed errors; (7) GARCH, (8) AGARCH and (9) GJR, all three with skewed Student's t-distributed errors; (10) NM-GARCH, (11) NM-AGARCH and (12) NM-GJR, all three NM-GARCH models with zero means in the mixture component densities; (13) NM-GARCH, (14) NM-AGARCH and (15) NM-GJR, all three general NM-GARCH models.

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TABLE 3

Estimation results for the DAX 30 index

Model	*(1)*	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*	*(13)*	*(14)*	*(15)*
p_1 [d.f. for				9.002572	9.537122	9.596825	9.392559	9.8487	9.9290	0.9660	0.9762	0.9680	0.9613	0.9701	0.9604
(4)–(9)]				(11.72)	(11.94)	(11.80)	(11.18)	(11.40)	(11.25)	(100.65)	(118.45)	(98.95)	(92.86)	(97.75)	(89.04)
μ_1 [skew for							−0.1056	−0.1064	−0.1069				0.0044	0.0041	0.0043
(7)–(9)]							(−4.41)	(−4.43)	(−4.46)				(2.29)	(2.12)	(2.19)
ω_1	8.8E–4	4.7E–4	8.9E–4	3.4E–4	1.4E–4	4.4E–4	3.5E–1	1.7E–4	4.6E–4	1.5E–4	3.2E–5	2.5E–4	1.4E–4	2.9E–5	2.3E–4
	(8.02)	(3.80)	(7.83)	(3.37)	(0.95)	(4.05)	(3.42)	(1.17)	(4.17)	(2.32)	(0.29)	(3.18)	(2.21)	(0.27)	(3.11)
α_1	0.0825	0.0668	0.0320	0.0789	0.0772	0.0413	0.0787	0.0776	0.0414	0.0608	0.0664	0.0363	0.0595	0.0640	0.0355
	(11.59)	(13.59)	(4.93)	(8.48)	(8.27)	(3.83)	(8.53)	(8.33)	(3.92)	(9.19)	(8.98)	(4.24)	(9.18)	(8.92)	(4.26)
λ_1		0.0764	0.0768		0.0629	0.0731		0.0628	0.0730		0.0633	0.0549		0.0639	0.0538
		(7.82)	(7.53)		(4.64)	(5.01)		(4.69)	(5.00)		(4.93)	(4.63)		(4.89)	(4.54)
β_1	0.8982	0.9135	0.9080	0.9155	0.9142	0.9121	0.9153	0.9137	0.9121	0.9282	0.9200	0.9216	0.9291	0.9220	0.9230
	(100.01)	(134.74)	(115.60)	(96.70)	(96.17)	(93.55)	(97.01)	(96.16)	(93.91)	(125.62)	(113.55)	(113.67)	(127.09)	(114.79)	(115.37)
p_2										0.0340	0.0238	0.0320	0.0387	0.0299	0.0396
μ_2													−0.1096	−0.1328	−0.1039
ω_2										0.0354	−0.0150	0.0394	0.0259	0.0335	0.0317
										(0.49)	(−0.04)	(0.48)	(0.59)	(0.39)	(0.59)
α_2										0.4442	0.2226	−0.1441	0.3941	0.3729	−0.1193
										(0.58)	(0.31)	(−0.31)	(0.70)	(0.48)	(−0.71)
λ_2											0.7468	0.8197		0.0470	0.7002
											(0.35)	(0.63)		(0.19)	(0.75)
β_2										0.7120	0.4451	0.7110	0.7345	0.7155	0.7155
										(1.30)	(0.84)	(1.18)	(1.75)	(1.04)	(1.48)
Unconditional σ	21.31%	20.86%	20.22%	24.85%	22.75%	20.97%	24.10%	23.31%	21.45%	21.83%	21.52%	20.11%	21.40%	21.71%	20.22%
Unconditional σ_1										20.60%	20.54%	18.98%	20.11%	20.59%	19.01%
Unconditional σ_2										44.30%	46.39%	41.64%	40.68%	42.70%	38.01%
Unconditional τ							−0.1517	−0.1467	−0.1464	−7.97E–6	−8.40E–6	−1.01E–5	−0.1678	−0.1697	−0.1673
Unconditional k	1.6567	1.1240	1.5100	—	12.6898	—	—	1.7023	—	5.1276	3.5368	3.0636	4.6434	3.9313	2.7148

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TABLE 3

(continued)

Model	*(1)*	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*	*(13)*	*(14)*	*(15)*
BIC	−0.4339	−0.4393	−0.4410	−0.4884	−0.4925	−0.4927	−0.4912	−0.4955	−0.4957	−0.4853	−0.4878	−0.4875	−0.4870	−0.4891	−0.4893
Moment tests 1%	2	2	3	2	1	2	0	1	1	3	2	2	2	2	2
Density	2.2739	2.3948	1.9538	1.4085	1.8255	2.4724	1.2864	1.4249	1.3516	1.0503	1.2268	2.1285	0.9681	1.0154	1.5470
ACF	0.1889	0.3591	0.3116	—	0.9764	—	—	0.0570	—	0.0584	0.1345	0.1800	0.0443	0.0620	0.2201

Notes: Results of estimating the 15 different GARCH models using daily returns on the equity indices for the period 1 January 1991 to 21 October 2005 are reported. The total number of observations is 3,733. The first row shows the degrees of freedom in the Student t-GARCH models, and the probability of the first component in the NM-GARCH models; the second row shows the skewness in the skewed Student t-GARCH models and the mean of the first component in the general NM-GARCH models. Thereafter, we show the model's coefficient estimates and, for the NM-GARCH models, the second component's probability and mean, and the coefficient estimates of the second component. Numbers in parenthesis represent t-values.

Below this we list the unconditional volatility of each model, and for each of the components in the NM-GARCH models; the unconditional skewness and kurtosis, if they exist ['—' indicates negative kurtosis, or an unreasonably high value for the autocorrelation function (ACF) statistic, meaning that in that model the fourth moment is not defined]. Finally, we list the results of the four diagnostic tests (a)–(d) described in section IV. These are: (a) the Bayesian information criterion (BIC); (b) the moment specification tests of Newey (1985) at 1% significance; (c) the modified Kolmogorov–Smirnov statistic for the fit between the unconditional density simulated using the model and that observed empirically, in the historical data; and (d) the mean-squared error criterion to assess the fit of the model's theoretical ACF to the empirical ACF.

The models are: (1) GARCH, (2) AGARCH and (3) GJR, all three with normally distributed errors; (4) GARCH, (5) AGARCH and (6) GJR, all three with symmetric Student's t-distributed errors; (7) GARCH, (8) AGARCH and (9) GJR, all three with skewed Student's t-distributed errors; (10) NM-GARCH, (11) NM-AGARCH and (12) NM-GJR, all three NM-GARCH models with zero means in the mixture component densities; (13) NM-GARCH, (14) NM-AGARCH and (15) NM-GJR, all three general NM-GARCH models.

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TABLE 4

Estimation results for the Eurostoxx 50 index

Model	*(1)*	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*	*(13)*	*(14)*	*(15)*
p_1 [d.f. for				8.1884	8.5490	8.5869	8.5988	8.8517	8.9319	0.9506	0.9567	0.9571	0.9444	0.9493	0.9490
(4)–(9)]				(10.91)	(10.77)	(10.86)	(10.33)	(10.22)	(10.27)	(69.16)	(82.20)	(78.95)	(64.15)	(66.80)	(67.90)
μ_1 [skew for							−0.1017	−0.1029	−0.1049				0.0057	0.0054	0.0055
(7)–(9)]							(−4.25)	(−4.31)	(−4.39)				(3.21)	(2.98)	(3.07)
ω_1	4.5*E*–4	1.6*E*–4	4.5*E*–4	2.2*E*–4	9.1*E*–5	2.7*E*–4	2.3*E*–4	1.1*E*–4	2.8*E*–4	1.1*E*–4	4.0*E*–5	1.6*E*–4	1.1*E*–4	5.4*E*–5	1.7*E*–4
	(8.23)	(1.88)	(8.30)	(3.34)	(0.95)	(3.89)	(3.38)	(1.18)	(4.01)	(2.30)	(0.60)	(2.95)	(2.26)	(0.80)	(3.03)
α_1	0.0691	0.0586	0.0299	0.0743	0.0720	0.0410	0.0752	0.0734	0.0419	0.0624	0.0661	0.0394	0.0605	0.0642	0.0384
	(11.79)	(13.24)	(4.98)	(8.77)	(8.52)	(4.02)	(8.89)	(8.61)	(4.19)	(9.37)	(9.26)	(4.64)	(9.17)	(8.97)	(4.55)
λ_1		0.0709	0.0630		0.0508	0.0623		0.0504	0.0635		0.0428	0.0510		0.0443	0.0520
		(7.70)	(6.97)		(4.24)	(4.61)		(4.32)	(4.70)		(4.01)	(4.57)		(4.14)	(4.66)
β_1	0.9167	0.9270	0.9230	0.9209	0.9208	0.9198	0.9198	0.9194	0.9185	0.9249	0.9208	0.9203	0.9261	0.9206	0.9196
	(131.78)	(168.25)	(147.83)	(109.75)	(110.01)	(108.43)	(109.38)	(108.94)	(107.88)	(124.03)	(119.86)	(117.61)	(121.98)	(112.35)	(110.74)
p_2										0.0494	0.0433	0.0429	0.0556	0.0507	0.0510
μ_2													−0.0967	−0.1018	−0.1030
ω_2										0.0140	−0.0055	0.0253	0.0098	0.0084	0.0089
										(0.85)	(−0.07)	(0.67)	(1.03)	(0.48)	(0.58)
α_2										0.3306	0.2710	−0.0625	0.3155	0.2288	0.1122
										(0.88)	(0.67)	(−0.20)	(1.05)	(0.64)	(0.26)
λ_2											0.4416	0.6205		0.0641	0.1900
											(0.77)	(0.60)		(0.39)	(0.69)
β_2										0.7860	0.4909	0.7008	0.7996	0.8332	0.8416
										(3.37)	(2.92)	(1.63)	(4.44)	(3.24)	(3.15)
Unconditional σ	17.89%	17.78%	17.02%	21.60%	19.57%	18.06%	21.17%	20.37%	18.80%	18.65%	18.26%	17.32%	18.21%	18.47%	17.64%
Unconditional σ_1										17.43%	17.27%	16.25%	16.91%	17.28%	16.43%
Unconditional σ_2										34.54%	33.29%	33.09%	31.77%	32.09%	31.09%
Unconditional τ							−0.1591	−0.1564	−0.1579	−1.05*E*-5	−8.26*E*-6	−1.20*E*-5	−0.2015	−0.1882	−0.2101
Unconditional k	1.5502	1.2574	1.5431	—	14.7870	—	—	3.7850	—	5.1022	3.2070	2.8952	4.7779	3.8641	3.3103

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TABLE 4

(continued)

Model	*(1)*	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*	*(13)*	*(14)*	*(15)*
BIC	−0.7481	−0.7553	−0.7547	−0.7979	−0.8014	−0.8015	−0.8008	−0.8045	−0.8047	−0.7895	−0.7910	−0.7912	−0.7927	−0.7936	−0.7943
Moment tests 1%	2	2	3	2	2	2	2	1	2	4	4	4	3	2	3
Density	2.1384	2.2230	1.8409	1.1173	1.5988	2.2568	1.0173	1.1309	1.0852	1.2707	1.2732	1.4062	0.7634	0.9386	0.7297
ACF	0.2479	0.4044	0.3427	—	0.8580	—	—	0.2339	—	0.0508	0.3303	0.3867	0.0720	0.1501	0.2778

Notes: Results of estimating the 15 different GARCH models using daily returns on the equity indices for the period 1 January 1991 to 21 October 2005 are reported. The total number of observations is 3,733. The first row shows the degrees of freedom in the Student t-GARCH models, and the probability of the first component in the NM-GARCH models; the second row shows the skewness in the skewed Student t-GARCH models and the mean of the first component in the general NM-GARCH models. Thereafter, we show the model's coefficient estimates and, for the NM-GARCH models, the second component's probability and mean, and the coefficient estimates of the second component. Numbers in parenthesis represent t-values.

Below this we list the unconditional volatility of each model, and for each of the components in the NM-GARCH models; the unconditional skewness and kurtosis, if they exist ['—' indicates negative kurtosis, or an unreasonably high value for the autocorrelation function (ACF) statistic, meaning that in that model the fourth moment is not defined]. Finally, we list the results of the four diagnostic tests (a)–(d) described in section IV. These are: (a) the Bayesian information criterion (BIC); (b) the moment specification tests of Newey (1985) at 1% significance; (c) the modified Kolmogorov–Smirnov statistic for the fit between the unconditional density simulated using the model and that observed empirically, in the historical data; and (d) the mean-squared error criterion to assess the fit of the model's theoretical ACF to the empirical ACF.

The models are: (1) GARCH, (2) AGARCH and (3) GJR, all three with normally distributed errors; (4) GARCH, (5) AGARCH and (6) GJR, all three with symmetric Student's t-distributed errors; (7) GARCH, (8) AGARCH and (9) GJR, all three with skewed Student's t-distributed errors; (10) NM-GARCH, (11) NM-AGARCH and (12) NM-GJR, all three NM-GARCH models with zero means in the mixture component densities; (13) NM-GARCH, (14) NM-AGARCH and (15) NM-GJR, all three general NM-GARCH models.

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TABLE 5

Estimation results for the FTSE 100 index

Model	*(1)*	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*	*(13)*	*(14)*	*(15)*
p_1 [d.f. for				13.7316	14.1768	14.0745	14.4429	14.6183	14.6046	0.9462	0.9598	0.9587	0.9288	0.9583	0.9564
(4)–(9)]				(6.47)	(6.67)	(6.74)	(6.42)	(6.59)	(6.69)	(32.97)	(42.96)	(40.14)	(1.00)	(42.80)	(38.70)
μ_1 [skew for							−0.0679	−0.0657	−0.0669				0.0049	0.0029	0.0036
(7)–(9)]							(−2.87)	(−2.80)	(−2.85)				(1.00)	(1.68)	(1.97)
ω_1	2.5E–4	−4.7E–5	2.4E–4	2.1E–4	−1.2E–4	2.1E–4	2.0E–4	−1.1E–4	2.1E–4	1.3E–4	−1.7E–4	1.6E–4	1.1E–4	−1.6E–4	1.6E–4
	(4.40)	(−0.54)	(5.39)	(3.35)	(−1.10)	(4.25)	(3.34)	(−1.02)	(4.26)	(2.52)	(−1.74)	(3.72)	(2.24)	(−1.65)	(3.78)
α_1	0.0734	0.0558	0.0155	0.0674	0.0547	0.0114	0.0674	0.0547	0.0122	0.0570	0.0493	0.0074	0.0532	0.0482	0.0069
	(10.94)	(8.54)	(2.42)	(8.09)	(7.23)	(1.61)	(8.19)	(7.32)	(1.75)	(7.47)	(7.25)	(1.25)	(7.17)	(7.26)	(1.21)
λ_1		0.0785	0.0781		0.0862	0.0820		0.0849	0.0810		0.0902	0.0756		0.0899	0.0738
		(6.39)	(8.96)		(5.63)	(7.28)		(5.66)	(7.26)		(6.07)	(7.72)		(6.03)	(7.73)
β_1	0.9171	0.9316	0.9350	0.9247	0.9331	0.9379	0.9247	0.9333	0.9379	0.9305	0.9348	0.9416	0.9344	0.9362	0.9432
	(118.21)	(129.34)	(137.35)	(101.89)	(112.72)	(121.93)	(103.05)	(114.24)	(122.91)	(109.31)	(118.63)	(133.69)	(1.00)	(121.85)	(137.95)
p_2										0.0538	0.0402	0.0413	0.0712	0.0417	0.0436
μ_2													−0.0632	−0.0670	−0.0796
ω_2										0.0101	0.0109	0.0113	0.0056	0.0076	0.0051
										(1.19)	(0.83)	(0.65)	(1.00)	(0.78)	(0.78)
α_2										0.7933	0.9230	0.9226	0.6722	0.9401	0.9888
										(2.12)	(1.70)	(1.82)	(1.00)	(1.88)	(1.94)
λ_2											−0.0109	−0.0159		−0.0192	−0.3350
											(−0.24)	(−0.02)		(−0.44)	(−0.54)
β_2										0.5785	0.5644	0.5786	0.6250	0.5821	0.6658
										(2.82)	(2.12)	(1.70)	(1.00)	(2.59)	(4.37)
Unconditional σ	16.24%	15.38%	15.05%	16.13%	15.17%	14.79%	15.95%	15.37%	15.01%	15.86%	15.21%	14.81%	15.72%	15.51%	15.24%
Unconditional σ_1										15.01%	14.49%	14.03%	14.74%	14.74%	14.39%
Unconditional σ_2										26.69%	27.26%	27.27%	24.36%	27.04%	26.89%
Unconditional τ							−0.0742	−0.0713	−0.0726	−7.26E–6	−8.67E–6	−1.13E–5	−0.1354	−0.1186	−0.1576
Unconditional k	3.9916	1.5162	3.0582	5.5489	2.7292	8.2826	—	—	—	7.8609	3.8048	3.6864	7.9404	4.3925	4.4979

continued overleaf

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TABLE 5

(continued)

Model	*(1)*	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*	*(13)*	*(14)*	*(15)*
BIC	−1.0583	−1.0689	−1.0704	−1.0687	−1.0794	−1.0812	−1.0685	−1.0791	−1.0810	−1.0638	−1.0738	−1.0760	−1.0657	−1.0750	−1.0777
Moment tests 1%	4	1	1	2	1	2	2	1	0	2	1	1	2	0	0
Density	60.7739	1.2624	0.6579	0.7213	1.2385	1.7367	1.0483	1.1607	0.9953	0.6626	1.0954	1.6242	0.8146	0.8484	0.8520
ACF	0.4478	0.0988	0.1832	0.8115	0.0976	0.6602	—	—	—	0.2547	0.1275	0.1529	0.2636	0.1087	0.1239

Notes: Results of estimating the 15 different GARCH models using daily returns on the equity indices for the period 1 January 1991 to 21 October 2005 are reported. The total number of observations is 3,733. The first row shows the degrees of freedom in the Student t-GARCH models, and the probability of the first component in the NM-GARCH models; the second row shows the skewness in the skewed Student t-GARCH models and the mean of the first component in the general NM-GARCH models. Thereafter, we show the model's coefficient estimates and, for the NM-GARCH models, the second component's probability and mean, and the coefficient estimates of the second component. Numbers in parenthesis represent t-values.

Below this we list the unconditional volatility of each model, and for each of the components in the NM-GARCH models; the unconditional skewness and kurtosis, if they exist ['—' indicates negative kurtosis, or an unreasonably high value for the autocorrelation function (ACF) statistic, meaning that in that model the fourth moment is not defined]. Finally, we list the results of the four diagnostic tests (a)–(d) described in section IV. These are: (a) the Bayesian information criterion (BIC); (b) the moment specification tests of Newey (1985) at 1% significance; (c) the modified Kolmogorov–Smirnov statistic for the fit between the unconditional density simulated using the model and that observed empirically, in the historical data; and (d) the mean-squared error criterion to assess the fit of the model's theoretical ACF to the empirical ACF.

The models are: (1) GARCH, (2) AGARCH and (3) GJR, all three with normally distributed errors; (4) GARCH, (5) AGARCH and (6) GJR, all three with symmetric Student's t-distributed errors; (7) GARCH, (8) AGARCH and (9) GJR, all three with skewed Student's t-distributed errors; (10) NM-GARCH, (11) NM-AGARCH and (12) NM-GJR, all three NM-GARCH models with zero means in the mixture component densities; (13) NM-GARCH, (14) NM-AGARCH and (15) NM-GJR, all three general NM-GARCH models.

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tables the first row reports the degrees of freedom for the *t*-GARCH models (4)–(9) and the weight of the first component in the NM-GARCH models for models (10)–(15). Similarly, the third row reports the skewness parameter for models (7)–(9) and the mean of the first normal density for the NM-GARCH models.

Ex-post state probabilities

Figure 1 presents the daily Eurostoxx index returns for the period January 2004–October 2005. It is expected that the *ex-post* state probabilities will indicate a jump from the first state (characterized by a lower variance) to the second state (characterized by a larger variance) when the absolute returns increase in magnitude – or, from the second state to the first one, when absolute returns decrease. Figure 2 shows that this is indeed the case – the graph presents the estimated conditional volatilities of the two components plotted against the time-varying *ex-post* probability of the first state. Note that when the returns have a sharp increase in absolute value after a relatively tranquil period, the *ex-post* probability of the first state decreases, suggesting a switch from the first, less volatile state to the second state. In such a situation both conditional volatilities show an upward jump. The volatilities exhibit a downward jump when the switch is from the second to the first states.

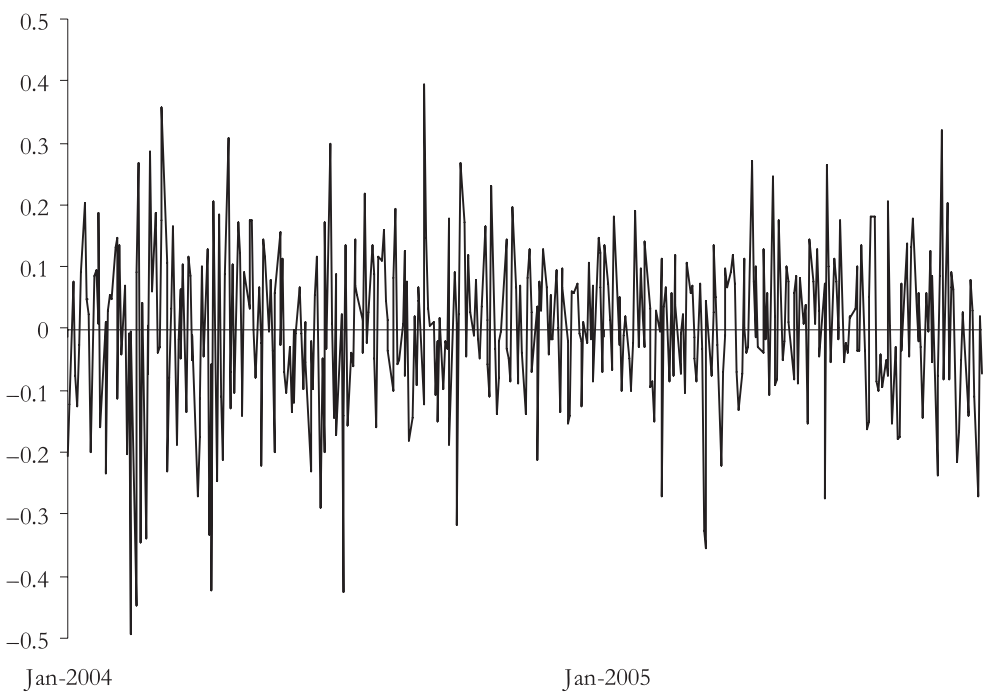

Figure 1. The returns on the Eurostoxx index

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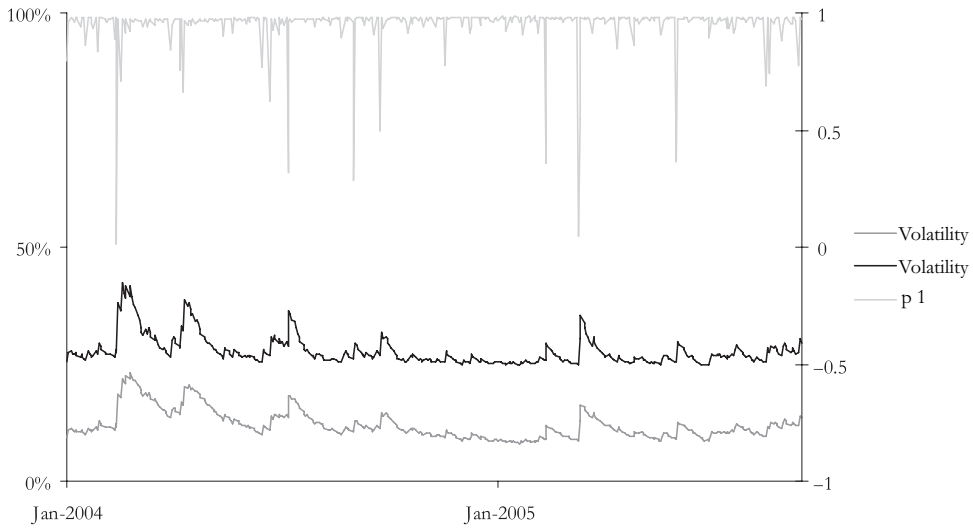

Figure 2. The conditional volatilities and the *ex-post* time-varying probability of the first state for the Eurostoxx index

Model selection

The following selection criteria are used:

(a) *Bayesian information criterion* (BIC): The model with the lowest BIC is chosen.

(b) *Moment specification tests*: Following Newey (1985), we test for normality in the standardized residuals, checking the first four moments and for zero autocorrelations in their first four powers, using a Wald test. Although this test has the disadvantage that it tends to reject the null too often in small samples (Godfrey, 1988), this test was used for GARCH-type processes by Nelson (1991), Harvey and Siddique (1999), and Brooks, Burke and Persand (2005) and a multivariate test was developed by Ding and Engle (2001).[20] We proceed with the transformation:

$$u_t = \Phi^{-1}\left(\sum_{i=1}^{K} p_i \Phi_i(\varepsilon_t)\right),$$

where Φ is the cumulative normal density function; if the model is the true data-generating process (DGP), then this transformation should lead to i.i.d. standard normal series.[21] We test a total of 20 conditions (the first four moments of u_t and the first four autocorrelations in its first four powers) and the test statistics

[20]Interestingly, the authors have shown, using Monte Carlo simulations, that the small sample properties of this test's statistics are good.

[21]A similar transformation was used by Haas *et al.* (2004a). This transformation is different from the one of Lanne and Saikkonen (2003), where the resulting series have unit variance and preserve autocorrelation, but are not i.i.d. normal.

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for the moment tests have a $\chi^2(1)$ distribution. The tables report the number of tests (out of 20) that are rejected at the 1% significance level.

(c) *Unconditional density fit*: The density test is on the histogram fit between the model-simulated data and the original data. This is one of the most difficult tests for models to pass as it tests for the unconditional distributional fit. The model returns are simulated[22] and their histogram is estimated using a non-parametric kernel approach. Several alternatives are available for the kernel, our chosen function being that of Epanechnikov (1969). Then the model selection criterion is based on the modified Kolmogorov–Smirnov (KS) statistic (Kolmogoroff, 1933; Smirnov, 1939; Massey, 1951; Khamis, 2000).

(d) *Autocorrelation function (ACF) analysis*: In contrast to (c) this test captures the dynamic properties of the model-squared returns – namely, the fit to the empirical autocorrelations of the squared returns. The Appendix states the theoretical ACFs of the different models and we apply the mean-squared error (MSE) criterion to assess the fit of the models.

(e) *Value-at-risk (VaR) analysis*: The VaR tests developed by Christoffersen (1998) are performed. Using the estimated parameters, the in-sample VaR estimates are computed. Based on these values, the occurrences of exceeding the VaR are counted. This is achieved by building several indicator functions. The first one equals zero except when a return lower than minus the VaR occurs, when it equals one:

$$I_{t;\alpha} = \begin{cases} 1, & r_t < -\mathrm{VaR}_{t|t-j;\alpha} \\ 0, & \text{otherwise} \end{cases}, \quad \text{where } \alpha \text{ is the level of significance.}$$

Then there are a set of four indicator functions:

$$J_{ij,t;\alpha} = \begin{cases} 1, & I_{t-1;\alpha} = i \quad \text{and} \quad I_{t;\alpha} = j \\ 0, & \text{otherwise} \end{cases}, \quad \text{for } i, j = 0, 1.$$

The three test statistics, for unconditional coverage, independence and conditional coverage are as follows:

$$LR_{uc;\alpha} = -2\log\left[\left(\frac{1-\alpha}{1-\pi_\alpha}\right)^{n_{0;\alpha}}\left(\frac{\alpha}{\pi_\alpha}\right)^{n_{1;\alpha}}\right] \sim \chi^2(1),$$

$$LR_{ind;\alpha} = -2\log\left[\left(\frac{1-\pi_{2;\alpha}}{1-\pi_{01;\alpha}}\right)^{n_{00;\alpha}}\left(\frac{\pi_{2;\alpha}}{\pi_{01;\alpha}}\right)^{n_{01;\alpha}}\left(\frac{1-\pi_{2;\alpha}}{1-\pi_{11;\alpha}}\right)^{n_{10;\alpha}}\left(\frac{\pi_{2;\alpha}}{\pi_{11;\alpha}}\right)^{n_{11;\alpha}}\right] \sim \chi^2(1),$$

$$LR_{cc;\alpha} = LR_{uc;\alpha} + LR_{ind;\alpha} \sim \chi^2(2),$$

[22] We simulate returns, based on the estimated parameters, and to ensure that the simulated density is not affected by small sample size we use 50,000 replications. Also, to avoid any influence of the starting values, each simulation has 1,000 steps ahead of time but we only use the last simulated return.

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$$\text{where } n_{1;\alpha} = \sum_{t=1}^{T} I_{t;\alpha}, n_{0;\alpha} = T - n_{0;\alpha}, \pi_\alpha = T^{-1} n_{1;\alpha}, n_{ij;\alpha} = \sum_{t=1}^{T} J_{ij,t;\alpha},$$

$$\pi_{01;\alpha} = n_{01;\alpha}(n_{00;\alpha} + n_{01;\alpha})^{-1}, \pi_{11;\alpha} = n_{11;\alpha}(n_{10;\alpha} + n_{11;\alpha})^{-1}, \pi_{2;\alpha} = T^{-1}(n_{01;\alpha} + n_{11;\alpha}).$$

For simplicity, only the first two test statistics are reported and analysed.

The results of these specification tests are shown in the last four rows of Tables 2–5 and in Table 6. These can be interpreted as follows:

(a) *BIC*: Most series favour the skewed t-AGARCH and skewed t-GJR models, except the FTSE for which the symmetric t-GJR model is preferable. According to the BIC, NM models are outperformed by t-GARCH models. Asymmetric NM-GARCH models fare better than the symmetric models, and also models with non-zero means have a lower BIC than NM models that assume zero means for the normal components. However, this is a very simple criterion and it ignores the models' ability to capture all the characteristics of the data.

(b) *Moment specification tests*: These tests show that the most basic models, that is, (1)–(3) do not capture the higher moments. But beyond this observation, the moment tests do not distinguish well between the models. We find that most models have several rejections for these tests and none of them perform consistently well according to this criterion.

(c) *Unconditional density fit*: This shows a clear preference for the NM-GARCH models (13), (14) and (15) (i.e. with different component means). We conclude that it is necessary to model both persistent and dynamic sources of asymmetry. One reason why the t-GARCH models (4)–(9) do not perform as well as the NM-GARCH models, according to this criterion, is that in the t-GARCH models the conditions for existence of the fourth moment are not always satisfied.[23] As a result, they often produce unconditional densities with unrealistic (negative) unconditional kurtosis, suggesting that we should exclude these instances from the test for that particular model. However, we require the time periods when the densities are computed to be consistent across all models, so we do not exclude these instances from the analysis.

(d) *ACF analysis*: This test also favours the asymmetric NM-GARCH models. Again we see that models based on the t distribution perform badly in many situations and quite often the ACF estimates of these models are negative. This is probably because these models do not capture the fourth moment. In contrast, all the NM models perform very well according to this criterion.

(e) *VaR analysis*: The independence and conditional coverage test results in Table 6 show that normal GARCH models are in general very unsuitable for VaR estimations. Additionally, restricted NM-GARCH models fail some of the VaR tests, again proving that they are not suitable for VaR calculations.

[23] These conditions are given in the supplementary Appendix to Alexander and Lazar (2006).

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TABLE 6

In-sample value-at-risk results

Test	*Significance*	*Data*	*(1)*	*(2)*	*(3)*	*(4)*	*(5)*	*(6)*	*(7)*	*(8)*	*(9)*	*(10)*	*(11)*	*(12)*	*(13)*	*(14)*	*(15)*
Independence	1%	CAC	7.4***	6.6**	7.4***	2.3	1.9	1.5	0.0	0.1	0.2	2.8*	5.2**	5.2**	0.0	0.8	
		DAX	5.9**	5.2**	3.9**	0.4	0.6	0.8	0.0	0.1	0.0	2.4	4.6**	4.6**	0.0	0.6	0.2
		FTSE	2.8*	2.3	1.5	1.1	0.0	0.2	0.0	0.2	0.3	1.5	1.1	0.6	0.0	0.1	0.0
		EUROS	14***	9.1***	9.1***	2.9*	0.6	0.6	0.0	0.0	0.0	6.7***	2.9*	4.6**	0.0	0.0	0.0
		CAC	9.0***	11***	17***	0.1	0.0	0.1	1.3	1.9	3.7*	0.9	5.8**	5.8**	0.0	0.0	
	0.5%	DAX	14***	11***	13***	2.0	0.9	1.4	0.2	0.2	0.0	0.9	2.6	2.6	2.7*	1.9	0.8
		FTSE	7.8***	4.0**	5.8**	1.4	0.3	0.9	0.0	0.0	0.0	2.6	0.9	1.9	0.0	0.1	0.2
		EUROS	19***	13***	15***	3.7*	3.0	3.0	1.7	0.2	0.8	4.5**	0.8	1.2	0.2	0.2	0.1
		CAC	7.2***	5.3**	7.2***	3.7*	2.3	2.3	0.2	1.2	0.4	1.0	0.2	0.2	1.0	1.0	
	0.1%	DAX	14***	9.3***	9.3***	2.3	0.4	0.4	1.2	0.4	0.0	0.2	1.2	1.2	1.0	1.0	0.2
		FTSE	14***	11***	14***	1.2	3.7*	5.3**	0.4	1.2	3.7*	1.2	1.2	1.2	0.4	0.0	0.4
		EUROS	41***	24***	31***	1.1	1.1	1.1	0.2	0.0	0.3	0.2	0.2	0.0	1.0	1.0	1.0
Conditional coverage	1%	CAC	0.0	0.1	1.6	0.2	1.1	1.1	0.8	0.8	0.9	1.3	1.5	1.5	0.7	1.0	
		DAX	4.0**	4.2**	1.8	3.0*	2.8*	2.7	3.7*	4.0**	0.8	2.1	4.4**	4.4**	3.5*	2.8*	0.6
		FTSE	1.2	1.2	1.1	1.0	0.7	0.9	0.8	0.7	0.6	1.1	1.0	1.0	0.8	0.8	0.7
		EUROS	0.7	1.1	1.1	2.0	2.7*	2.7*	3.6*	0.7	0.7	1.3	1.3	0.1	0.8	0.8	0.8
		CAC	0.6	0.7	0.8	0.2	0.2	0.2	0.1	0.1	0.1	0.3	0.5	0.5	0.2	0.2	
	0.5%	DAX	3.7*	0.9	0.8	6.5**	2.3	2.1	3.4*	3.4*	3.0*	0.3	0.4	0.4	0.1	0.1	0.1
		FTSE	0.6	0.4	0.5	0.3	0.2	0.3	0.2	0.2	0.2	0.4	0.3	0.3	0.2	0.2	0.2
		EUROS	3.0*	0.7	0.7	1.6	0.4	0.4	0.3	0.2	0.3	0.4	0.3	0.3	0.2	0.2	0.2
		CAC	0.1	0.0	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	
	0.1%	DAX	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
		FTSE	0.1	0.1	0.1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
		EUROS	0.3	0.2	0.2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

Notes: The models are: (1) GARCH, (2) AGARCH and (3) GJR, all three with normally distributed errors; (4) GARCH, (5) AGARCH and (6) GJR, all three with symmetric Student's t-distributed errors; (7) GARCH, (8) AGARCH and (9) GJR, all three with skewed Student's t-distributed errors; (10) NM-GARCH, (11) NM-AGARCH and (12) NM-GJR, all three NM-GARCH models with zero means in the mixture component densities; (13) NM-GARCH, (14) NM-AGARCH and (15) NM-GJR, all three general NM-GARCH models. The reported test statistics follow a $\chi^2(1)$ distribution. *, ** and *** indicate values significant at 10%, 5% and 1% significance levels, respectively. NM, normal mixture; GARCH, generalized autoregressive conditional heteroscedasticity.

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Symmetric and skewed t-GARCH models fail only few of the tests but the NM-AGARCH and NM-GJR models achieve the best results, passing both tests for all indices.

In summary, the normal GARCH model is the worst fit by all criteria and the Student's t-GARCH models only fit well according to the BIC criteria, which is very basic. Often t-GARCH models do not capture the fourth moment of the returns, and perform badly according to both the density fit and the ACF criteria. The NM models with different component means fit better than those with identical means, and the asymmetric components with the additional leverage effect improve the fit even further. Our results indicate that a two-state asymmetric NM-GARCH model with two distinct sources of asymmetry, that is, persistent and dynamic, is superior to all other models considered. For the dynamic asymmetry (i.e. the leverage effect), it does not really matter which asymmetric specification is used: the NM-AGARCH and NM-GJR formulations appear to work equally well.

Regime-specific volatility behaviour

Each of the two-state asymmetric NM-GARCH models captures a lower-volatility component that occurs with a high probability (the stable component) and a high-volatility component with a very low probability (the crash component). Table 7 summarizes the characteristics of these volatility components in each of the four European stock markets during the sample period considered.[24] In each market the stable component is characterized by a high associated probability, a positive mean

TABLE 7

Summary of regime-specific behaviour in volatility

	CAC	*DAX*	*Eurostoxx*	*FTSE*
Stable component				
Probability	0.978	0.970	0.958	0.949
Annualized mean return	7.9%	6.5%	4.6%	8.5%
Unconditional volatility	21%	21%	15%	17%
Volatility reaction (α_1)	0.0524	0.064	0.0482	0.0642
Volatility persistence (β_1)	0.9311	0.922	0.9362	0.9206
Leverage effect (λ_1)	0.1058	0.0639	0.0899	0.0443
Crash component				
Probability	0.022	0.030	0.042	0.051
Annualized mean return	−353%	−210%	−106%	−161%
Unconditional volatility	44%	43%	27%	32%
Volatility reaction (α_2)	1.5497	0.3729	0.9401	0.2288
Volatility persistence (β_2)	0.6172	0.7155	0.5821	0.8332
Leverage effect (λ_2)	−0.0311	0.0470	−0.0192	0.0641

[24]The results in this table are based on the NM-AGARCH model but similar results are obtained from the NM-GJR model. To obtain the annualized mean returns we multiply the mean estimates by $\sqrt{250}$.

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return and a low volatility. The stable mean return is lowest in the FTSE, at only 4.6% per annum but the unconditional volatility is also low: at 15%, it is the lowest of all. Notably the Eurostoxx index has the highest return (8.5%) and the second lowest volatility of the four markets (17%). On the other hand, the CAC and DAX indices have higher volatility, around 21%, but their mean return is less than that of Eurostoxx during their stable state. However, they have been in their stable regime more often than the FTSE and Eurostoxx: for the latter, a stable regime occurred only about 95% of the time between 1991 and 2005. In the stable regime the least reactive and most persistent volatility is that of the FTSE series; the Eurostoxx and DAX display slightly more reaction and less persistence than the other indices; and the leverage effect is most pronounced in the CAC index.

The crash market regime occurred between about 2% and 5% of the time, depending on the index. The CAC and the DAX have the highest unconditional crash volatility (over 40%) and the lowest expected return during a crash (about −350%, on an annualized basis, for the CAC). This is probably because the CAC and DAX indices are less liquid than the FTSE and Eurostoxx. In our sample the crash mean returns are around −100% for the FTSE and −160% for the Eurostoxx. By comparison, during the 1987 crisis the FTSE lost over 400%, on an annualized basis. All indices, and the FTSE and CAC in particular, are highly reactive to market shocks in the crash regime, yet because the persistence parameters are all low, especially on the FTSE, the effect of a shock soon dies out. The crash leverage effects are similar to the stable leverage effects for the DAX and Eurostoxx, but the highly reactive FTSE and CAC indices have an inverse leverage effect in a crash regime. This indicates that a positive return will lead to higher volatility than a negative return, but only during the crash regime. A possible explanation of this is that investors anticipate further falls after a modest recovery.

V. Equity index implied volatility skews

Out-of-the-money put options on an equity index are an attractive form of insurance for investors that fear a general market decline; and with index option market makers in relatively short supply, the market prices of these options are often far higher than the Black–Scholes (1973) model prices based on the at-the-money volatility. Consequently, the implied volatility of these options is commonly found to be higher than the implied volatility of at-the-money call and put options and out-of-the-money calls. This leads to a skew (or 'smirk') in equity index implied volatility that has been highly pronounced since the global stock market crash in 1987, as shown by Bates (1991), Rubinstein (1994), Jackwerth and Rubinstein (1996), Derman and Kamal (1997), Tompkins (2001) and many others. An equity index implied volatility skew is associated with a negatively skewed implied risk neutral stock returns density.

Since the global crash of 1987 the skewness in risk neutral stock index densities has, in general, been much greater than the skewness estimated from historical data on stock index returns. The difference between the physical and risk neutral skews

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has been the subject of extensive academic research, as in Bates (1997, 2000), Bakshi, Kapadia and Madan (2003) and many others. These papers argue that a time-varying volatility risk aversion adjustment is necessary to reconcile the physical and risk-neutral distributions of equity indices. The volatility risk aversion can be recovered using empirical pricing kernels based either on unconditional historical returns, as in Ait-Sahalia and Lo (2000) and Jackwerth (2000) or on conditional returns densities, as in Rosenberg and Engle (2002). This led Bates (2003) to argue that the difference between the risk-neutral and real-world distributions can only be explained by the existence of a time-varying volatility risk premium. However, he excluded the possibility of time-varying skewness and kurtosis, as higher moments are constant in a single-state GARCH process.

The asymmetric NM-GARCH process provides a model with time-varying conditional skewness and kurtosis that are endogenous to the model. Because of this (and unlike single-state GARCH models) the volatility skew will exhibit a term structure even in the physical measure. Moreover, an uncertainty over two possible volatility states, each of which exhibits volatility clustering but with quite different characteristics, provides a strong justification for a time-varying volatility risk premium.

A natural question to ask, therefore, is how these properties are reflected in the equity skews implied by the GARCH process. In this section, we compare the implied volatility skews in the physical measure generated by asymmetric NM-GARCH models with those implied by other GARCH models. As we shall see, leverage effects are clearly very important. In all markets we find very pronounced skews for the NM-GARCH models, even in the absence of a risk premium, and the skew persistence that is captured by the difference in the means of the variance components is only of secondary importance.

For illustration, we use the parameter estimates of the FTSE index returns given in Table 4 on a representative selection of 5 of the 15 models: the normal GARCH, asymmetric t-GARCH, asymmetric t-GJR, NM-GARCH and NM-GJR models. We simulate volatility skew surfaces using each of these five models and compare their characteristics. Starting with $S_0 = 100$ and using $r = 0.03$, we simulate the dynamics of the index value as:

$$S_t = S_{t-\Delta t} \exp((r - \sigma_t^2/2)\Delta t + \varepsilon_t \sqrt{\Delta t}).$$

Then, for a fixed strike K and maturity T the time zero price of a European call option is computed as $\exp(-rT)E(\max(0, S_T - K))$. Simulating 50,000 times and computing the average call value gives the estimate of the option price. Then, applying the inverse Black–Scholes formula gives the simulated implied volatility at (K, T). We take a range of strikes between 80 and 130 and a range of maturities from 3 to 18 months. The results are shown in Figure 3.

We have used the same vertical scale from 10% to 25% volatility for each smile, as this makes the comparison easier. The first two skews, from the normal and skewed t-GARCH models are unrealistic. The volatility level is too low and there is no evidence of a negative skew even for the skewed t-GARCH model. The GJR skew in

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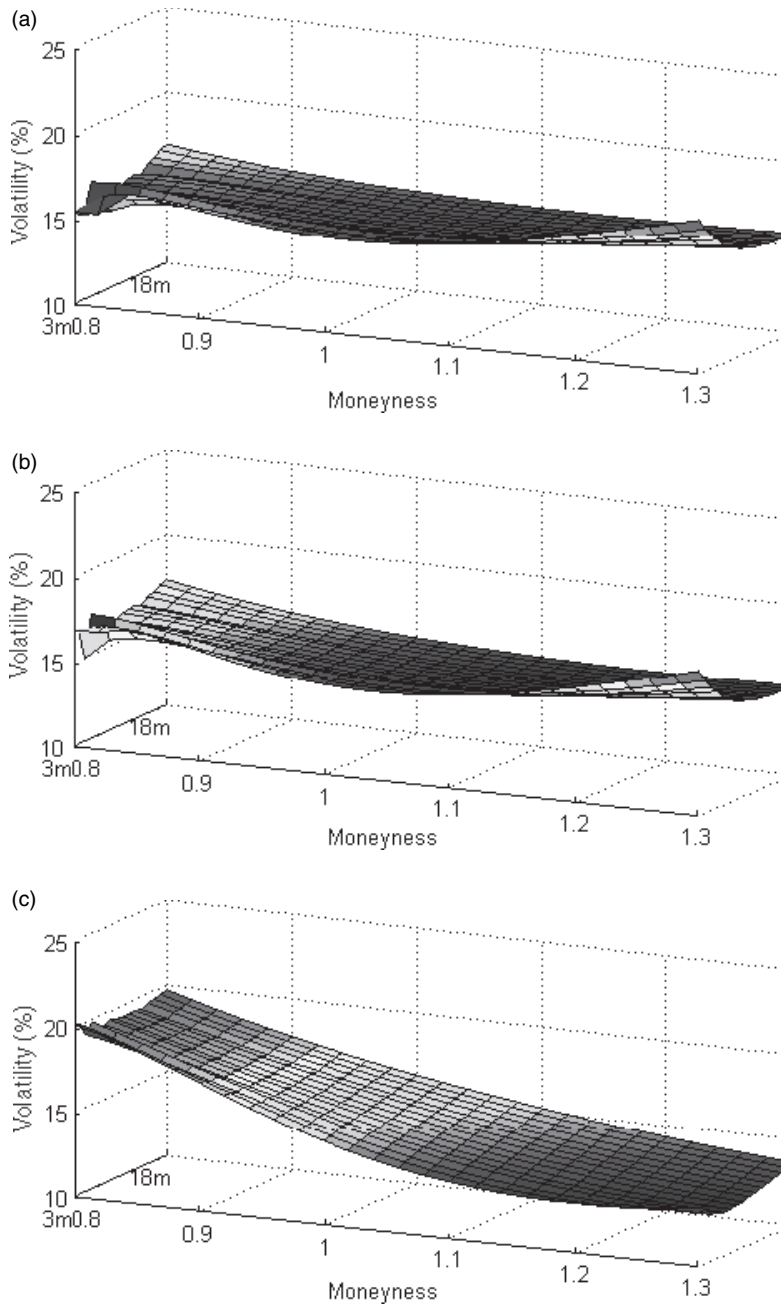

Figure 3. Simulated FTSE index skews up to 3 months. (a) GARCH; (b) GARCH with skewed *t*-distribution; (c) GJR with skewed *t*-distribution; (d) NM-GARCH; (e) NM-GJR
Notes: Implied volatility skews for call options are simulated using 50,000 runs. Zero interest rate is assumed. The parameters for the simulation are taken from Table 3. NM, normal mixture; GARCH, generalized autoregressive conditional heteroscedasticity

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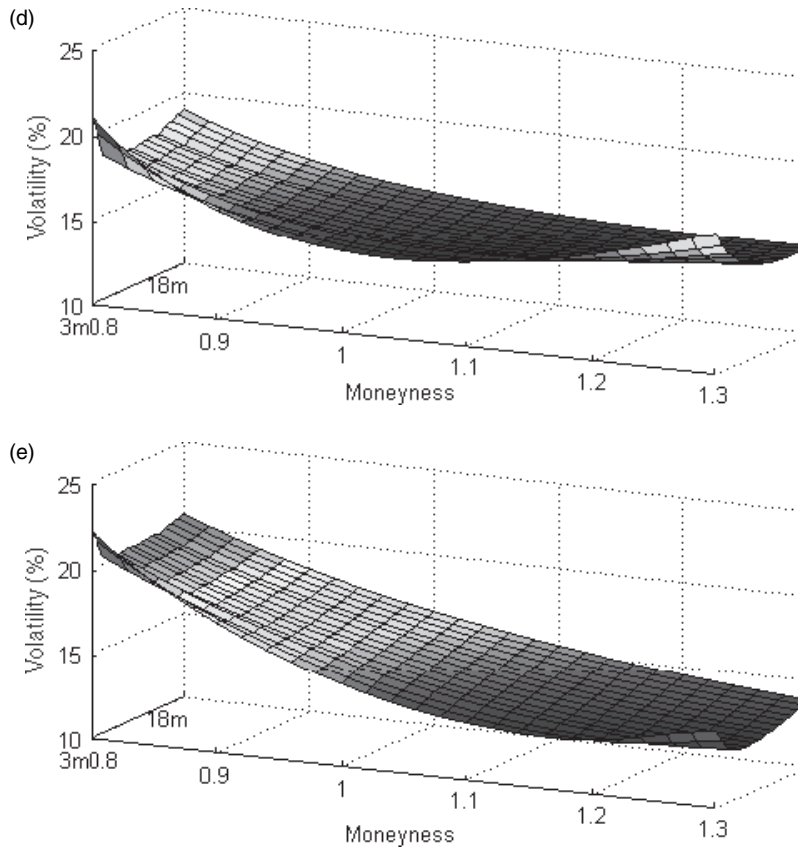

Figure 3. *(continued)*

Figure 3c is more realistic, with substantially higher volatility for in-the-money (ITM) calls than out-of-the-money (OTM) calls. Still, there is little evidence of a volatility term structure as, for a given moneyness, volatility is almost constant as maturity increases. The NM-GARCH without leverage effect has more of a term structure and the short-term skew is not exactly linear with respect to the moneyness, a feature that is quite common in index markets. Finally, the NM-GJR model produces the most realistic skew: it is more pronounced and less linear than in the single-component GJR model, and there is more variation of volatility over time. We conclude that including non-zero means in the components of the mixture provides a skew that persists into longer-dated options but it is the leverage effect that primarily determines the slope of the skew for short-dated options.

VI. Summary and conclusions

The majority of the GARCH models specify a single time-varying volatility state and thus offer only one scenario for market behaviour. Also their standardized

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higher-moment specification is not realistic: time variation in conditional skewness and kurtosis is ignored (except in a few instances where it is specified exogenously to the GARCH model). This paper considers the NM-GARCH model with two volatility states, which has endogenous time-varying conditional higher moments. We introduce an additional asymmetry as a result of leverage effects, so that the model is suitable for capturing regime-specific stock return volatility.

We first ask whether this additional source of asymmetry is necessary, given that NM-GARCH models with different mean returns already exhibit time-varying conditional skewness and kurtosis (in contrast to single-state GARCH models). The answer to this question is undoubtedly yes. Both the statistical model-selection criteria and the simulations of the volatility skew justify the addition of the leverage effect. The state-dependent expected return already captures non-zero skewness, but only in the unconditional returns distribution. The addition of dynamic asymmetry, via the state-dependent leverage effect appears to be very important, because it dramatically improves the time-series fit of the NM-GARCH models.

Empirical results on four European stock indices compare the fit of symmetric and asymmetric NM-GARCH models with single-state normal and symmetric and skewed t-GARCH specifications. The overall conclusion from applying five statistical criteria to 15 fitted models is a clear superiority of asymmetric NM-GARCH models. These models are then used to characterize the regime-specific behaviour of the stock indices. Between 1991 and 2005, the relative frequency of a crash regime varied from about 2% for the CAC to about 5% for the Eurostoxx. In the stable regime the indices have broadly similar behaviour (except that leverage effects are strongest in the CAC and weakest in the Eurostoxx) but the crash regime behaviour is more diverse. The CAC index displays the most extreme crash behaviour, returning -350% in annual terms with an unconditional crash volatility of 44%; in contrast, the FTSE returned only -100% and the unconditional crash volatility is far lower. All indices are highly sensitive to market shocks in the crash regime, particularly the CAC, but persistence in volatility is very low, indicating that volatility quickly returns to more normal levels. The leverage effect in the CAC and the FTSE is positive (but small) during a crash period, indicating that investors might be anticipating further falls after a modest recovery. Although the Eurostoxx has the highest crash probability, in the stable regime it has the highest mean return (8.5%) and a low unconditional volatility (17%), although this is not as low as the FTSE volatility. As the crash behaviour of the Eurostoxx is also the least extreme, apart from the FTSE, it may be the safest investment alternative of the indices considered.

If agents' beliefs about the future can be informed by past behaviour, the rich structure that is revealed by estimating a two-state asymmetric NM-GARCH model could provide invaluable information for investors, traders, risk managers, regulators, governments and policy-makers. For instance, in risk management stress tests are commonly applied using *ad hoc* rules, such as changing volatilities to 100% without associating any probability to this event. The NM-GARCH model that we have implemented in this paper has the potential to provide objective, probabilistic,

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regime-specific stress tests on stock index volatility, which is one of the main risk factors for both equity and credit portfolios. This is clearly an interesting subject for further research. Finally, we have demonstrated that single-state GARCH models imply unrealistic shapes for the implied volatility skew surfaces for stock indices. In contrast, asymmetric NM-GARCH models provide a rich behavioural structure that can match all the observed characteristics of implied volatility skew surfaces. This demonstrates their superiority over single-state GARCH models, which must resort to assuming that volatility risk premium are time-varying, to explain the observed skew characteristics. Asymmetric NM-GARCH models can generate realistic term structures for volatility skews, even with a zero volatility risk premium. This indicates that option pricing based on two-state asymmetric GARCH models is likely to be a fruitful area for further research.

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Appendix: Theoretical properties of asymmetric NM-GARCH models

This Appendix details the theoretical properties of the asymmetric NM-GARCH models that are considered in this paper. Section A1 derives the conditional skewness and kurtosis of the models. These are used in section A2 to provide conditions, which are imposed on the model estimation, for the existence of the conditional variance and the finiteness of the conditional skewness. Section A3 provides analytic derivatives for the conditional variance with respect to the model parameters. The use of analytic derivatives enhances both the speed and quality of the model estimation. Section A4 derives the theoretical ACF of the models, which are used in our empirical study to determine the model's goodness-of-fit.

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A1. Moments

Nelson (1990) gave necessary and sufficient conditions for the existence of the moments of the normal GARCH (1,1) model, proving that for the normal GARCH model given by the equation

$$\sigma_t^2 = \omega + (\alpha\eta_{t-1}^2 + \beta)\sigma_{t-1}^2$$

the pth moment of the error term exists iff

$$E[(\alpha\eta_{t-1}^2 + \beta)^{p/2}] < 1.$$

For example, when we assume that $\alpha = 0.09875$ and $\beta = 0.9$ then the second moment exists, but the third and higher moments do not.

Unfortunately, no such necessary and sufficient conditions exist for the NM-GARCH model, and in particular, the conditions for the existence of the skewness and kurtosis are unknown. Of course, the following moment formulae only apply if the moment exists. We use the following notation:

$$x = E(\varepsilon_t^2) = E(\sigma_t^2) \text{ and } y_i = E(\sigma_{it}^2) \text{ for } i = 1, \ldots, K.$$

Taking expectations of equations (3) and (6) yields the result for NM-AGARCH and taking expectations of equations (4) and (6) yields the result for the NM-GJR model. We obtain:

$$x = E(\varepsilon_t^2) = E(\sigma_t^2) = \left(\sum_{i=1}^{K} p_i\mu_i^2 + \sum_{i=1}^{K} \frac{p_i\omega_i^*}{(1-\beta_i)}\right)\left(1 - \sum_{i=1}^{K} \frac{p_i\delta_i}{(1-\beta_i)}\right)^{-1},$$

$$y_i = (1-\beta_i)^{-1}(\omega_i^* + \delta_i x), \quad i = 1, \ldots, K$$

where

$$\omega_i^* = \begin{cases} \omega_i + \alpha_i\lambda_i^2 & \text{for NM-AGARCH} \\ \omega_i & \text{for NM-GJR} \end{cases},$$

$$\delta_i = \begin{cases} \alpha_i & \text{for NM-AGARCH} \\ \alpha_i + \lambda_i E(d_t^-)(1+\rho) & \text{for NM-GJR} \end{cases},$$

and ρ is the correlation between d_t^- and ε_t^2.[25]

Under the assumption that higher moments exist and are finite, the third moment is

$$h = E(\varepsilon_t^3) = \sum_{i=1}^{K} p_i E_i(\varepsilon_t^3) = \sum_{i=1}^{K} p_i\mu_i(3y_i + \mu_i^2)$$

[25] In our empirical application, we have approximated $E(d_t^-)$ by 0.5 and ϱ by 0.

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and the skewness can be expressed as $s = hx^{-3/2}$. The excess kurtosis in both models is:

$$\kappa = \frac{E(\varepsilon_t^4)}{E(\varepsilon_t^2)^2} - 3 = \frac{z}{x^2} - 3, \quad \text{where } z = E(\varepsilon_t^4) = \frac{3\mathbf{p}'\mathbf{B}^{-1}\mathbf{f} - s}{1 - 3\mathbf{p}'\mathbf{B}^{-1}\mathbf{g}}.$$

The fourth moment uses the following notation:

$$\mathbf{p} = (p_1, \ldots, p_K)', \quad s = \sum_{i=1}^{K} p_i(6\mu_i^2 y_i^2 + \mu_i^4),$$

$$\mathbf{B} = (b_{ij}) = \begin{bmatrix} 1 - \beta_1^2 - 2\delta_1\beta_1 e_{11} & -2\delta_1\beta_1 e_{12} & \ldots & -2\delta_1\beta_1 e_{1K} \\ -2\delta_2\beta_2 e_{21} & 1 - \beta_2^2 - 2\delta_2\beta_2 e_{22} & \ldots & -2\delta_K\beta_K e_{2K} \\ \vdots & \vdots & \vdots & \vdots \\ -2\delta_K\beta_K e_{K1} & -2\delta_K\beta_K e_{K2} & \ldots & 1 - \beta_K^2 - 2\delta_K\beta_K e_{KK} \end{bmatrix},$$

where $e_{ij} = a_{ij}p_j$ and

$$\mathbf{A} = (a_{ij}) = \begin{bmatrix} 1 - \sum_{\substack{k=1 \\ k\neq 1}}^{K} \frac{p_k\beta_1\delta_k}{1-\beta_1\beta_k} & -\frac{p_2\delta_1\beta_2}{1-\beta_1\beta_2} & \ldots & -\frac{p_K\delta_1\beta_K}{1-\beta_1\beta_K} \\ -\frac{p_1\delta_2\beta_1}{1-\beta_2\beta_1} & 1 - \sum_{\substack{k=1 \\ k\neq 2}}^{K} \frac{p_k\beta_2\delta_k}{1-\beta_2\beta_k} & \ldots & -\frac{p_K\delta_2\beta_K}{1-\beta_2\beta_K} \\ \vdots & \vdots & \vdots & \vdots \\ -\frac{p_1\delta_K\beta_1}{1-\beta_K\beta_1} & -\frac{p_2\delta_K\beta_2}{1-\beta_K\beta_2} & \ldots & 1 - \sum_{\substack{k=1 \\ k\neq K}}^{K} \frac{p_k\beta_K\delta_k}{1-\beta_K\beta_k} \end{bmatrix},$$

$$\mathbf{g} = \begin{pmatrix} \gamma_1 + 2\delta_1\beta_1 d_1 \\ \vdots \\ \gamma_K + 2\delta_K\beta_K d_K \end{pmatrix} \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} w_1 + 2\delta_1\beta_1 c_1 \\ \vdots \\ w_K + 2\delta_K\beta_K c_K \end{pmatrix}.$$

Also

$$d_i = \sum_{j=1}^{K} a_{ij} \left(\sum_{\substack{k=1 \\ k\neq j}}^{K} \frac{p_k\delta_j\delta_k}{1-\beta_j\beta_k} \right) \quad \text{and} \quad \gamma_i = \begin{cases} \alpha_i^2 & \text{for NM-AGARCH} \\ \alpha_i^2 + \lambda_i\delta_i & \text{for NM-GJR} \end{cases}.$$

Furthermore

$$w_i = \begin{cases} \omega_i^2 + x(2\omega_i\alpha_i + 6\alpha_i^2\lambda_i^2) + \alpha_i^2(\lambda_i^4 - 4\lambda_i h) \\ \quad + 2\omega_i\alpha_i\lambda_i^2 + 2(\omega_i + \alpha_i\lambda_i^2)\beta_i y_i & \text{for NM-AGARCH} \\ \omega_i^2 + 2\omega_i\delta_i x + 2\omega_i\beta_i y & \text{for NM-GJR} \end{cases},$$

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$$c_i = \sum_{j=1}^{K} a_{ij} \left[\left(\sum_{\substack{k=1 \\ k \neq j}}^{K} \frac{p_k r_{jk}}{1 - \beta_j \beta_k} \right) + y_j q \right] \quad \text{with } q = \sum_{k=1}^{K} p_k \mu_k^2,$$

$$r_{ik} = \begin{cases} \omega_i \omega_k + x[(\omega_i \alpha_k + \omega_k \alpha_i) + \alpha_i \alpha_k (\lambda_i^2 + 4\lambda_i \lambda_k + \lambda_k^2)] \\ \quad -2\alpha_i \alpha_k h(\lambda_i + \lambda_k) \\ \quad + \beta_i y_i (\omega_k + \alpha_k \lambda_k^2) + \beta_k y_k (\omega_i + \alpha_i \lambda_i^2) \\ \quad + \omega_i \alpha_k \lambda_k^2 + \omega_k \alpha_i \lambda_i^2 + \alpha_i \alpha_k \lambda_i^2 \lambda_k^2 & \text{for NM-AGARCH} \\ \quad \omega_i \omega_k + x(\omega_i \delta_k + \omega_k \delta_i) + \beta_i y_i \omega_k + \beta_k y_k \omega_i & \text{for NM-GJR.} \end{cases}$$

A2. Parameter constraints

We require the following set of conditions for the existence and non-negativity of variance and the finiteness of the third moment in both the NM-AGARCH and NM-GJR models:

$$m = \sum_{i=1}^{K} p_i \mu_i^2 + \sum_{i=1}^{K} \frac{p_i \omega_i^*}{(1 - \beta_i)} > 0, \quad n = \sum_{i=1}^{K} \frac{p_i (1 - \delta_i - \beta_i)}{(1 - \beta_i)} > 0 \text{ and } \omega_i^* + \delta_i \frac{m}{n} > 0.$$

Also, we must have that

$$0 < p_i < 1, \quad i = 1, \ldots, K-1, \quad \sum_{i=1}^{K-1} p_i < 1, \quad 0 < \alpha_i, \quad 0 \leq \beta_i < 1.$$

These conditions are required for the existence and positivity of the overall variance and individual variances as well. However, these conditions do not guarantee the existence of the third moment; they only assure that, if the third moment exists, then it will be finite. No straightforward parameter constraint exists for the existence and finiteness of the fourth moment; so we simply put $0 < E(\varepsilon_t^4) < \infty$.

A3. Analytic derivatives of the asymmetric NM-GARCH models

The only difference from the NM-GARCH model derivatives given by Alexander and Lazar (2006) is the first- and second-order derivatives of σ_{it}^2 with respect to $\boldsymbol{\gamma}_i$ and these are as follows:

$$\frac{\partial \sigma_{it}^2}{\partial \boldsymbol{\gamma}_i} = \mathbf{z}_{it} + \beta_i \frac{\partial \sigma_{it-1}^2}{\partial \boldsymbol{\gamma}_i}, \quad \frac{\partial^2 \sigma_{it}^2}{\partial \boldsymbol{\gamma}_i \partial \boldsymbol{\gamma}_i'} = w_{it} + \beta_i \frac{\partial^2 \sigma_{it-1}^2}{\partial \boldsymbol{\gamma}_i \partial \boldsymbol{\gamma}_i'} \text{ with } w_{it} = A_{it} + A_{it}^T.$$

NM-AGARCH:
$\mathbf{z}_{it} = (1, (\varepsilon_{t-1} - \lambda_i)^2, -2\alpha_i(\varepsilon_{t-1} - \lambda_i), \sigma_{it-1}^2)'$. The starting values ($t = 0$) are:

$$\frac{\partial \sigma_{i0}^2}{\partial \boldsymbol{\gamma}_i} = (1, s^2 + \lambda_i^2, 2\alpha_i \lambda_i, s^2)', \quad \text{where } s^2 = T^{-1} \sum_{t=1}^{T} \varepsilon_t^2, \text{ and}$$

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$$A_{it} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2(\varepsilon_{t-1} - \lambda_i) & \alpha_i & 0 \\ & \left(\dfrac{\partial \sigma^2_{it-1}}{\partial \boldsymbol{\gamma}_i}\right)' & & \end{bmatrix} \quad \text{with } \frac{\partial^2 \sigma^2_{i0}}{\partial \boldsymbol{\gamma}_i \, \partial \boldsymbol{\gamma}'} = (1-\beta_i)^{-1}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2\lambda_i & s^2 + \lambda_i^2 \\ 0 & 2\lambda_i & 2\alpha_i & 2\alpha_i \lambda_i \\ 1 & s^2 + \lambda_i^2 & 2\alpha_i \lambda_i & 2s^2 \end{bmatrix}.$$

NM-GJR:
$z_{it} = (1, \varepsilon^2_{t-1}, d^-_{t-1}\varepsilon^2_{t-1}, \sigma^2_{it-1})'$. The starting values $t = 0$ are $\frac{\partial \sigma^2_{i0}}{\partial \gamma_i} = (1, s^2, 0.5s^2, s^2)'$, and

$$A_{it} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ & \left(\dfrac{\partial \sigma^2_{it-1}}{\partial \boldsymbol{\gamma}_i}\right)' & & \end{bmatrix} \quad \text{with } \frac{\partial^2 \sigma^2_{i0}}{\partial \boldsymbol{\gamma}_i \, \partial \boldsymbol{\gamma}'_i} = \frac{1}{(1-\beta_i)} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & s^2 \\ 0 & 0 & 0 & 0.5s^2 \\ 1 & s^2 & 0.5s^2 & 2s^2 \end{bmatrix}.$$

A4. ACF of the squared errors in the asymmetric NM-GARCH models[26]

The following formulae for the ACF only hold if the fourth moment exists.

$$\rho_k = \mathrm{Corr}(\varepsilon^2_t, \varepsilon^2_{t-k}) = \frac{\mathrm{Cov}(\varepsilon^2_t, \varepsilon^2_{t-k})}{\mathrm{Var}(\varepsilon^2_t)} = \frac{E[\varepsilon^2_t \varepsilon^2_{t-k}] - x^2}{E[\varepsilon^4_t] - x^2} = \frac{c_k - x^2}{z - x^2},$$

where

$$c_k = E[\varepsilon^2_t \varepsilon^2_{t-k}] = x \sum_{i=1}^{K} p_i \mu_i^2 + \sum_{i=1}^{K} p_i E[\sigma^2_{it} \varepsilon^2_{t-k}] = x \sum_{i=1}^{K} p_i \mu_i^2 + \sum_{i=1}^{K} p_i b_{ik}.$$

NM-AGARCH:

$$b_{ik} = (\omega_i + \alpha_i \lambda_i^2) x + \alpha_i c_{k-1} + \beta_i b_{ik-1}, \quad k > 1$$
$$b_{i1} = (\omega_i + \alpha_i \lambda_i^2) x + \alpha_i c_0 + \beta_i b_{i0} + -2\alpha_i \lambda_i h, \quad k = 1.$$

NM-GJR:

$$b_{ik} = \omega_i x + (\alpha_i + 0.5\lambda_i) c_{k-1} + \beta_i b_{ik-1}.$$

The starting values are: $c_0 = z$ and $b_{i0} = c_i + d_i z + \mathbf{e}_i{}' \mathbf{B}^{-1}(\mathbf{f} + \mathbf{g}z)$.

[26]As the variance of the NM(K)-GARCH(1, 1) model can be expressed as a GARCH(K, K) variance, according to Bollerslev (1986), the autocorrelations can also be written as an AR(K) process.

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