J. Opl Res. Soc. Vol. 45, No. 8, pp. 954–958
Printed in Great Britain.
0160-5682/94 $9.00+0.00

The Constrained Nash Bargaining Solution

C. O. ALEXANDER and W. LEDERMANN
University of Sussex

We prove a simple condition which guarantees the existence and uniqueness of the constrained generalized Nash bargaining solution in $\mathbb{R}^2$. Our result is illustrated by a constant elasticity example of firm/union negotiations.

Key words: bargaining set, constrained Nash bargain, convexity

INTRODUCTION

Applications of the Nash bargaining solution to economic modelling may assume that, whilst the players' utility functions are over vectors $\boldsymbol{x} \in \mathbb{R}^n$, negotiations concern only one of these variables, each of the rest being set unilaterally. A common example of this is wage bargaining between firm and union, where employment is set by the firm according to its labour demand schedule. This type of model has been used in the empirical study by Nickell and Andrews[1] and in the theoretical work of Dowrick[2,3], Anderson and Devereux[4] and Hoel[5], to name only the most recent literature in this field. Existence of a unique constrained Nash bargaining solution has been assumed in this literature, but the results of Alexander and Ledermann[6] show that non-uniqueness may be a problem even in very general circumstances.

When firm/union negotiations are so constrained, the bargaining set reduces to a one-dimensional compact curve Γ within the original bargaining set. So the uniqueness conditions of Nash[7], which refer only to a two-dimensional compact bargaining set, cannot be used. In this note we prove that, in the case of constrained bargaining with $n = 2$, a unique generalized Nash bargaining solution exists provided that the Pareto optimal segment of Γ is 'convex upwards'. The shape of Γ is determined by the form of utility function chosen for each of the players, and so our uniqueness condition may be translated into conditions on players' utility functions.

Consider a segment of a curve Γ which is represented by $y = g(x)$, $x \in [r,\ s]$ where $0 \le r < s$ (see Figure 1). We wish to solve the problem

$$\max_{\Gamma} f(x,y)$$

where $f(x,y) = x^{\beta}y$, $\beta > 0$ and $y = g(x)$. On Γ the function f reduces to

$$F(x) = f(x,g(x)) = x^{\beta}g(x).$$

We now prove the following.

Theorem

There exists a unique $x_N \in [r,\ s]$ $(0 \le r < s)$ such that $\max_{\Gamma} f = F(x_N)$ if $g(x)$ satisfies the following conditions:

$$\text{(i) } g'(x) < 0 \quad \text{in} \quad [r,\ s];$$

and

$$\text{(ii) } g''(x) < 0 \quad \text{in} \quad [r,\ s].$$

Correspondence: C. O. Alexander, School of Mathematical and Physical Sciences, Mathematics Subject Group, University of Sussex, Falmer, Brighton BN1 9QH, UK

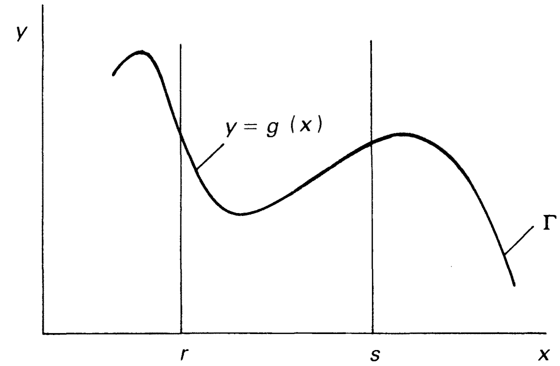

FIG. 1. *The bargaining curve (general case).*

Proof

We have that

$$F'(x) = \beta x^{\beta-1}g(x) + x^{\beta}g'(x).$$

Since we are looking for the maximum of F we may assume that $x \neq 0$. First we shall show that $F'(x) = 0$ cannot have more than one positive root in $[r, s]$. Suppose that

$$F'(x_1) = F'(x_2) = 0$$

and

$$0 < x_1 < x_2 \leq s,$$

that is

$$\beta x_1^{\beta-1}g(x_1) + x_1^{\beta}g'(x_1) = 0$$
$$\beta x_2^{\beta-1}g(x_2) + x_2^{\beta}g'(x_2) = 0,$$

or

$$\beta g(x_1) + x_1 g'(x_1) = 0$$
$$\beta g(x_2) + x_2 g'(x_2) = 0.$$

By Rolle's theorem there exists a number ξ in $[r, s]$ such that

$$\frac{\mathrm{d}}{\mathrm{d}x}\{\beta g(x) + xg'(x)\}_{x=\xi} = 0,$$

that is

$$\beta g'(\xi) + \xi g''(\xi) + g'(\xi) = 0,$$

or

$$(\beta + 1)g'(\xi) + \xi g''(\xi) = 0.$$

But this equation is impossible because, by virtue of conditions (i) and (ii), the left-hand side is strictly negative. Thus $F'(x) = 0$ has at most one root. We have to distinguish two cases.

(i) $F'(x) = 0$ has no root; in other words F' does not change its sign in $[r, s]$. If $F' > 0$, then F is monotonic increasing and $x_N = s$; if $F' < 0$, then F is monotonic decreasing and $x_N = r$.

(ii) There exists x_N such that $r < x_N < s$ and $F'(x_N) = 0$; we still have to show that x_N does correspond to a maximum. In fact we shall prove that $F''(x_N) < 0$. It is found that

$$F''(x) = \beta(\beta - 1)x^{\beta-2}g(x) + 2\beta x^{\beta-1}g'(x) + x^{\beta}g''(x).$$

Hence

$$F''(x_N) = \beta(\beta - 1)x_N^{\beta-2}g(x_N) + 2\beta x_N^{\beta-1}g'(x_N) + x_N^{\beta}g''(x_N), \qquad (1)$$

but

$$F'(x_N) = 0 = \beta x_N^{\beta-1}g(x_N) + x_N^{\beta}g'(x_N)$$

so that

$$\beta g(x_N) = -x_N g'(x_N).$$

Substituting in the first term on the right of (1) we obtain that

$$F''(x_N) = -(\beta - 1)x_N^{\beta-1}g'(x_N) + 2\beta x_N^{\beta-1}g'(x_N) + x_N^{\beta}g''(x_N),$$
$$= (\beta + 1)x_N^{\beta-1}g'(x_N) + x_N^{\beta}g''(x_N),$$

which is clearly negative, since both terms on the right are negative. Hence x_N does indeed correspond to a maximum.

Note that standard necessary conditions for the existence of a unique solution to the problem

$$\max_{\Gamma} f(x,y) \qquad \text{such that} \qquad y = g(x)$$

require indirect concavity of both f and g. The conditions of our theorem are far less restrictive, since they require only that a segment of the bargaining curve for a given range of x be 'convex upwards'.

We illustrate the theorem with an example which uses constant elasticity functional forms (a standard economic practice).This construction has been employed by Alexander[8] for investigation of the Kalai-Smorodinsky wage bargaining solution.

Suppose the union's utility over wages $\omega \in \mathbb{R}$ and employment $L \in \mathbb{R}$ is represented by the function

$$V(\omega, L) = Lu(\omega)$$

where the utility u: $\mathbb{R} \to \mathbb{R}$ has been normalized so that $u(\omega^*) = 0$ at the competitive wage ω^*. We assume further that

$$u(\omega) = \begin{cases} (1/a)(\omega - \omega^*)^a & \text{for } \omega \geqslant \omega^* \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < a < 1$ so that union members have constant risk aversion. The firm's utility is its profit

$$\Pi(\omega, L) = R(L) - \omega L$$

where the revenue function, namely

$$R(L) = (1/b)L^b$$

where $0 < b < 1$, exhibits constant wage elasticity of employment. The constrained Nash problem is

$$\max_{\omega} V(\omega, L)^{\beta}\, \Pi(\omega, L) \qquad \text{subject to} \qquad \omega = L^{b-1}, \qquad \omega \geq \omega^*$$

and our theorem tells us that a unique Nash solution exists provided that

$$\frac{\mathrm{d}\Pi}{\mathrm{d}V} < 0 \qquad \text{and} \qquad \frac{\mathrm{d}^2\Pi}{\mathrm{d}V^2} < 0$$

along the Pareto optimal boundary of the constraint curve Γ. Now Γ is the image of the labour demand curve $\omega = L^{b-1}$ in (V, Π) space, and Γ may be parameterized by ω as follows:

$$V = (\omega^{1/(b-1)}/a)(\omega - \omega^*)^a$$

and

$$\Pi = ((1 - b)/b)\omega^{b/(b-1)},$$

thus

$$\mathrm{d}\Pi/\mathrm{d}V = a(1 - b)\omega(\omega - \omega^*)^{1-a}(c\omega - \omega^*)^{-1}$$

where $c = 1 - a + ab$ (so $0 < c < 1$). Thus, Γ is horizontal when $\omega = \omega^*$ and $\mathrm{d}\Pi/\mathrm{d}V < 0$ along Γ up to a point where $\omega = \omega^*/c$. Assuming the feasible set of wages is sufficiently large, Γ is vertical when $\omega = \omega^*/c$, and when $\omega > \omega^*/c$, Γ bends back on itself as illustrated in Figure 2. Thus, the Pareto optimal section of Γ is parameterized by ω for $\omega \in [\omega^*, \omega^*/c)$ and clearly on this section, $\mathrm{d}\Pi/\mathrm{d}V < 0$.

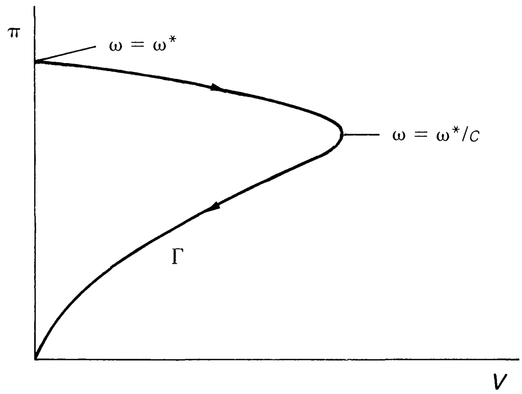

FIG. 2. *The bargaining curve for constant elasticity firm/union negotiations.*

Now

$$\frac{\mathrm{d}^2\Pi}{\mathrm{d}V^2} = \frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{\mathrm{d}\Pi}{\mathrm{d}V}\right)\Big/\frac{\mathrm{d}V}{\mathrm{d}\omega}$$

and since $\mathrm{d}V/\mathrm{d}\omega > 0$ when $\omega \in [\omega^*, \omega^*/c)$ we know that $\mathrm{d}^2\Pi/\mathrm{d}V^2 < 0$ if and only if $\mathrm{d}/\mathrm{d}\omega(\mathrm{d}\Pi/\mathrm{d}V) < 0$. Since

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\frac{\mathrm{d}\Pi}{\mathrm{d}V}\right) = a(1 - b)(c\omega - \omega^*)^{-2}(\omega - \omega^*)^{-a}M$$

where $M = (\omega - \omega^*)(c\omega - \omega^*) + \omega(1 - a)(c\omega - \omega^*) - c\omega(\omega - \omega^*)$, and clearly $M < 0$, so also $\mathrm{d}/\mathrm{d}\omega(\mathrm{d}\Pi/\mathrm{d}V)$, < 0 when $\omega \in [\omega^*, \omega^*/c)$.

The conditions of our theorem, namely that the Pareto optimal section of Γ be 'convex upwards', have been verified. Hence, the existence of a unique generalized Nash wage bargaining solution is proved for this constant elasticity example. Its value depends on the parameters of the problem; a, b, ω^* and β. For example, when $a = \frac{1}{2}$, $b = \frac{1}{3}$ and $\beta = 1$ (for symmetric bargaining), the wage that is agreed at the constrained Nash solution is

$$\omega_N = 4\omega^*/3.$$

REFERENCES

1. S. J. NICKELL and M. ANDREWS (1983) Unions, real wages and employment in Britain, 1951–79. *Oxford Economic Papers* **35**, 183–206.
2. S. DOWRICK (1989) Union-oligopoly bargaining. *Economic J.* **99**, 1123–1142.
3. S. DOWRICK (1990) The relative profitability of Nash bargaining on the labour demand curve or the contract curve. *Economics Lett.* **33**, 121–125.

4. S. P. ANDERSON and M. DEVEREUX (1989) Profit sharing and optimal labour contracts. *Canad. J. Economics* **22**, 425–433.
5. M. HOEL (1990) Local versus central wage bargaining with endogenous investments. *Scand. J. Econ.* **92**, 453–469.
6. C. O. ALEXANDER and W. LEDERMANN (1992) Bargaining sets and bargaining solutions for firm/union negotiations. University of Sussex MRR-92-7.
7. J. F. NASH (1950) The bargaining problem. *Econometrica* **18**, 155–162.
8. C. O. ALEXANDER (1992) The Kalai-Smorodinsky bargaining solution in wage negotiations. *J. Opl Res. Soc.* **43**, 779- 786.