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The Kalai–Smorodinsky Bargaining Solution in Wage Negotiations

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This paper characterizes the Kalai–Smorodinsky bargaining solution when firms and unions negotiate over wages alone, and firms set the level of employment in order to maximize profits given the agreed wage. The Kalai–Smorodinsky solution is analysed for the case that the wage elasticity of employment and the union's risk aversion are both constant. In this case there is a simple relationship between the Kalai–Smorodinsky and the Nash solutions.

Key words: industrial relations, manpower planning, game theory

AXIOMATIC MODELS OF BARGAINING BETWEEN A FIRM AND A UNION

This paper has been written for Steven Vajda and is offered to him with heartfelt thanks and great respect. It concerns models of the outcome of negotiations between a firm and a union. The utility functions of each party will depend on both the wage $\omega \in \mathbb{R}$ and the level of employment within the firm, $L \in \mathbb{R}$. Specifically, the union has utility function $U(\omega, L) = Lu(\omega)$, where $u(\cdot)$ denotes a standard continuous utility function of each of its (homogeneous) employed members, and this will be normalized so that union members unemployed with this firm receive a competitive wage ω^* and have utility zero. We assume that $u: \mathbb{R} \to \mathbb{R}$ is concave and $u'(\omega) > 0$. The firm has utility function $\Pi(\omega, L)$ given by the excess of revenue over labour costs, that is, the firm's profit. So

$$\Pi(\omega, L) = R(L) - \omega L$$

where the revenue function $R: \mathbb{R} \to \mathbb{R}$ is concave, $R(0) = 0$ and $R'(L) > 0$.

In the event that negotiations do not result in agreement we assume that no union members will be employed by the firm and that the firm makes zero profit, so the status quo point is $s = (0, 0)$. This is a special point within the *bargaining set* $B \subseteq \mathbb{R}$, which is given by

$$B = \{U(\omega, L), \Pi(\omega, L) \mid \omega^* \leqslant \omega \leqslant \bar{\omega}, 0 \leqslant L \leqslant \bar{L}\}.$$

We assume that both wages and employment have a maximum, $\bar{\omega}$ and $\bar{L}$ respectively. Then B is the continuous image of a compact set in (ω, L) space, so B will be a compact set in (Π, U) space. Now, in axiomatic bargaining theory it is usually assumed that the bargaining set be convex, and then a unique bargaining solution may exist. However, the firm–union bargaining set B may well be non-convex (for a detailed analysis of this bargaining set see McDonald and Solow[1]. Nevertheless, under the stated assumptions for $u(\cdot)$ and $R(\cdot)$, Alexander and Ledermann[2] show that the efficient boundary of B will be 'well behaved' enough for uniqueness of the standard bargaining solutions.

What is a *bargaining solution*? In the axiomatic approach to bargaining, the outcome of negotiations between two 'players' such as that just described, determines a unique point in the bargaining set which is characterized by an *arbitration scheme* $f: B \times s \to B$. It is assumed that the arbitration scheme satisfies certain axioms, the first of these axioms being that $f(B, s)$ is feasible, individually rational (that is, it ascribes both players at least as much as their status quo utility value) and Pareto optimal. Two other common axioms are symmetry:

$$f(T(B), T(s)) = T(f(B, s)) \text{ where } T(x, y) = (y, x)$$

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and invariance under affine transformations:

$$f(A(B), A(s)) = A(f(B, s)) \text{ for any affine transformation } A.$$

One other axiom is necessary if $f(B, s)$ is to determine a unique point in B, and such a point is called a 'bargaining solution'. We now review, briefly, the controversy surrounding the choice of the final bargaining axiom.

THE NASH BARGAINING SOLUTION

The standard model of firm/union negotiations which is used in pure and applied labour economics is the arbitration scheme proposed by John Nash[3] in 1950, despite substantial theoretical criticism of the *independence of irrelevant alternatives* axiom on which it is based. This axiom states that the Nash solution depends only on the status quo point s and the outcome selected, and not on any other points in the bargaining set B. That is, the Nash arbitration scheme $f: B \times s \to B$ is such that

$$f(B, s) = f(B', s) \; \forall B' \subseteq B \ni f(B, s) \in B'.$$

The main criticism of this axiom may be illustrated by the two bargaining sets in Figures 1 and 2, which yield the same Nash solution even though player 2 can get more in the first case, and which is thought to be counter-intuitive or, in some sense, 'unfair'—see Luce and Raiffa[4].

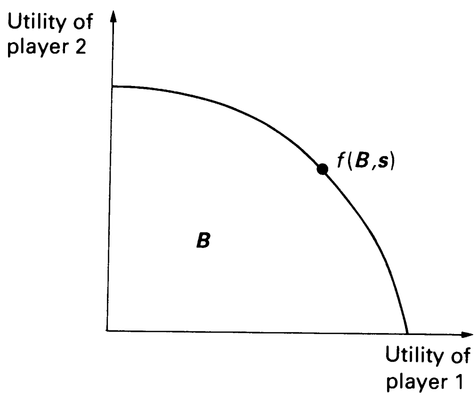

FIG. 1. *Bargaining set 1.*

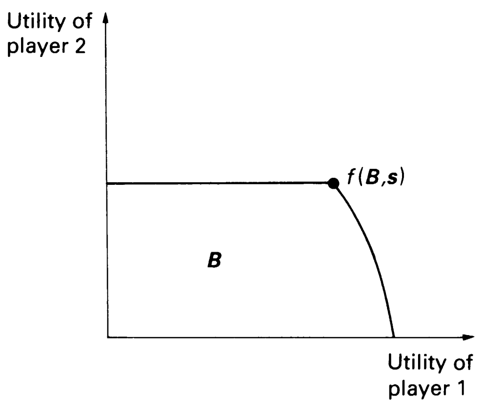

FIG. 2. *Bargaining set 2.*

Laboratory controlled experiments by Siegal and Fouraker[5], Crott[6], Nydegger and Owen[7] and Roth and Malouf[8] also suggest that the Nash solution is an unreasonable model of pairwise negotiations. The main body of evidence supports the view that players make interpersonal comparison of utility gains, such as would be the case under the 'Equal Gains' model of Myerson[9] or the 'Proportional Solution' of Kalai[10], but which cannot occur with the Nash solution because of the independence of irrelevant alternatives axiom.

Yet, to this day, Nash's arbitration scheme is almost always used in economic modelling. Examples include de Menil[11], Nickell and Andrews[12], Anderson and Devereux[13], Dowrick[14,15], Hoel[16], Leslie[17], and many others.

So why is the Nash solution so popular in labour economics? For one thing, it is perhaps the most mathematically tractable solution to employ. Secondly, there are several well-known strategic models of the negotiation process which implement Nash's axiomatic solution. Zeuthen[18] modelled the bargaining process as a sequence of small concessions by the player who incurs the smaller relative loss by so doing. However, there are substantial theoretical problems with this model, in particular the relative loss incurred by *total* capitulation is used to determine which player makes the concession, yet the model assumes that only a small concession is actually made. Consequently, Harsanyi[19,20] modified the Zeuthen model so that negotiations end after only two stages, with one player conceding totally in stage two to his opponent's demand in stage one. But the most realistic strategic model of bargaining which implements the Nash solution is that of Rubinstein[21]. Here, there is a sequence of offers and counter-offers, with a discount rate which motivates the parties to agree. Although the negotiation process is potentially infinite, Rubinstein showed that there is a unique perfect equilibrium which will be agreed upon at the first stage. Subsequently Binmore *et al.*[22] showed that the Nash solution corresponds to the unique perfect equilibrium partition in this model.

THE KALAI-SMORODINSKY SOLUTION

There is a strategic model of Moulin[23] which is less well known than Rubinstein's game, and which implements not the Nash solution but a certain type of proportional solution. In this solution, the contentious independence of the irrelevant alternatives axiom is replaced by a monotonicity axiom which states that if the bargaining set is enlarged so that for every pay-off to player i the pay-off to player j is increased, then player j's pay-off in the enlarged game should be no less than his pay-off in the original game. Replacing the independence of irrelevant alternatives by this monotonicity axiom allows players to make interpersonal comparison of utility gains, and hence is more in accord with the experimental evidence surveyed above.

The solution we refer to is named after Kalai and Smorodinsky[24] who showed that, if the bargaining set is compact and convex, and if there is at least one point $\boldsymbol{x} \in \boldsymbol{B}$ which is strictly individually rational for both players, these axioms are satisfied by a unique point on its boundary. This point, $\boldsymbol{k}$, and its relationship with the Nash solution $\boldsymbol{n}$, is illustrated in Figure 3.

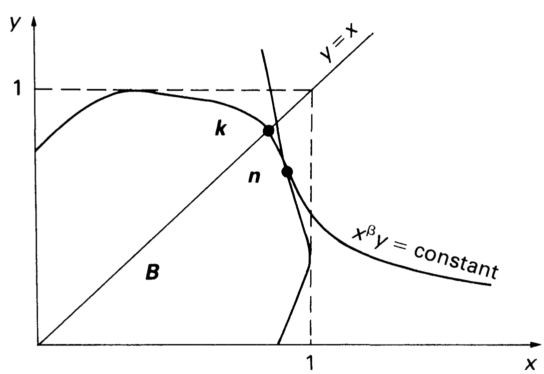

FIG. 3. *Kalai-Smorodinsky solution.*

In the figure x denotes the utility of player 1 and y the utility of player 2, both normalized so that the status quo is $(0, 0)$ and the *aspiration level* (i.e. the maximum utility in $\boldsymbol{B}$) of each party is unity. This graphic representation was the original characterization of the solution given by Kalai and Smorodinsky[24], namely the optimal intersection of the bargaining set with the line joining the status quo point with the aspiration point. Later Moulin[25] defined $\boldsymbol{k}$ as the solution to the problem

$$\max_{x,\,y} \min(x, y) \quad \text{such that} \quad (x, y) \in \boldsymbol{B}.$$

When $\boldsymbol{B}$ is a compact convex subset of the Euclidian plane the two representations are equivalent. Alexander and Ledermann[2] show that 'NE-regularity' of $\boldsymbol{B}$ is sufficient for their equivalence.

THE KALAI-SMORODINSKY SOLUTION ON A COMPACT CURVE IN $\boldsymbol{B}$

The Kalai-Smorodinsky solution has been largely ignored by labour economists, possibly because the mathematics involved are more complicated than for the Nash solution. The classic paper of McDonald and Solow[1] analyses the Nash solution and the Kalai-Smorodinsky solution in the case where negotiations concern both wages and the level of employment. But it may be more reasonable to assume that firms reserve the 'right to manage' employment. That is, they bargain over wages but set employment unilaterally to maximize profit given the agreed wage.

This paper characterizes the Kalai-Smorodinsky solution in a situation where firms and unions negotiate over wages only, the level of employment being determined unilaterally by profit maximization. Thus the condition

$$R'(L) = \omega$$

acts as a constraint on the wage bargaining problem.

The introduction of the constraint changes the problem. Utility maximization is no longer on a compact convex *set* in $\mathbb{R}^2$ but a compact *curve*. Figure 4 illustrates the point. Earlier we covered the individually rational part of the bargaining set corresponding to the positive quadrant in (L, ω) space. Note that x and y are normalized utilities so that both players have aspiration levels of unity. That is

$$x = U/U' \quad \text{and} \quad y = \Pi/\Pi'$$

where U' and Π' are the maximum values of U and Π respectively in $\boldsymbol{B}$. We call this normalized, individually rational bargaining set S. Now the labour demand curve $R'(L) = \omega$ in (L, ω) space is mapped onto a curve within this bargaining set in (x, y) space. We denote the curve by Γ, and it is indicated in bold.

There is now a new bargaining set for the problem, the original status quo point (at the origin)

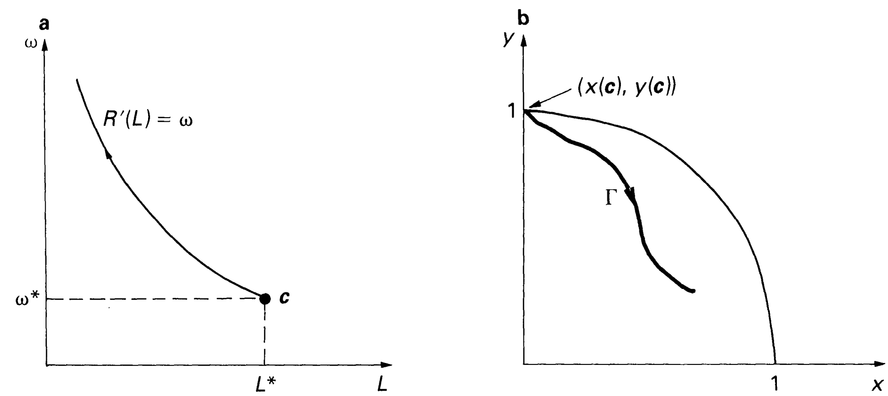

FIG. 4. *Utility maximization.*

plus the compact curve Γ, which need not be concave. However if we now normalize utilities so that

$$X = U/U^* \quad \text{and} \quad Y = \Pi/\Pi^*$$

where U^* and Π^* are the maximum values of U and Π respectively on Γ, the constrained Kalai–Smorodinsky solution is defined as the solution to the problem

$$\max_{X,\,Y} \min(X(\omega, L), Y(\omega, L))$$

such that

$$(X, Y) \in S$$

and

$$R'(L) = \omega.$$

The solution to this problem is not necessarily unique, and it may not coincide with the optimal intersection of Γ with the line $Y = X$, as the example in Figure 5 demonstrates.

Thus, characterization of the Kalai–Smorodinsky solution depends on the nature of the constraint curve Γ in (X, Y) space. Alexander and Ledermann[2] show that Γ is 'NE-regular' under standard assumptions on $u: \mathbb{R} \to \mathbb{R}$ and for general concave $R: \mathbb{R} \to \mathbb{R}$. A specific form of revenue function, for which the wage elasticity of employment is constant, is considered later in this paper.

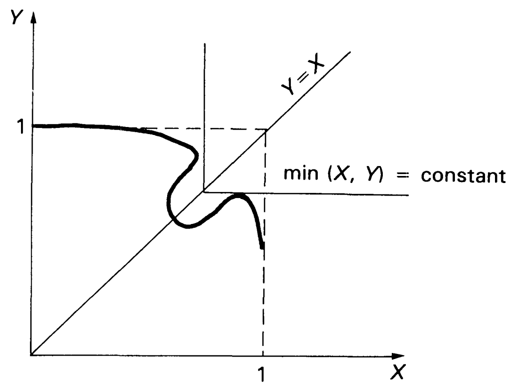

FIG. 5. *Kalai–Smorodinsky solution.*

COMPARISON OF THE CONSTRAINED KALAI-SMORODINSKY SOLUTION WITH THE CONSTRAINED NASH SOLUTION

In Alexander and Ledermann[2] it is shown that, under quite reasonable assumptions for $R(\cdot)$ and $u(\cdot)$ the Kalai–Smorodinsky solution $\boldsymbol{k}$ is indeed given by the intersection of the constraint curve Γ with the line $X = Y$. In this case it has a simple characterization and, moreover, bears a straightforward relationship with the generalized Nash solution, $\boldsymbol{n}$: in (L, ω) space we find $\boldsymbol{k}$ as the intersection of the labour demand curve $R'(L) = \omega$ with the curve

$$Lu(\omega)/U^* = (R(L) - \omega L)/\Pi^*.$$

The two curves are likely to intersect, because the labour demand curve always has a negative derivative (by concavity of R) and the other curve has derivative

$$\mathrm{d}\omega/\mathrm{d}L = (u(\omega) + A(R'(L) - \omega))/L(u'(\omega) + U^*)$$

where $A = (U^*/\Pi^*)$. So $\mathrm{d}\omega/\mathrm{d}L > 0$ when $R'(L) > \omega$. At a point of intersection of the two curves we have

$$u(\omega) = A((R(L)/L) - R'(L)).$$

Hence the above characterizes the Kalai–Smorodinsky solution.

To find $\boldsymbol{n}$, we work in (X, Y) space and maximize $\beta lnX + lnY$ over Γ (where β denotes the usual index of bargaining power). This yields the condition

$$u(\omega) = B((R(L)/L) - R'(L))$$

where $B = (U^*/\Pi^*)(\beta/\tau)$ and τ is the absolute value of the slope of the common tangent.

Hence, both $\boldsymbol{k}$ and $\boldsymbol{n}$ are characterized by the utility of the wage being proportional to the difference between the average product of labour and the marginal product of labour! (This is always non-negative assuming that $R(\cdot)$ is concave.) It is worth noting that McDonald and Solow[1] obtained a similar condition for the Nash solution to the unconstrained problem, where bargaining concerns both wages and employment. The condition there was that the wage be equal to the average of the marginal and average products of labour.

AN EXAMPLE

To gain more insight into the nature of the constraint curve $R'(L) = \omega$ in (X, Y) space we consider a specific example. Let

$$u(\omega) = (1/a)(\omega - \omega^*)^a$$

and

$$R(L) = (1/b)L^b$$

where $0 < a < 1$ and $0 < b < 1$. Thus, union members have constant risk aversion

$$-U'/((\omega - \omega^*)U'') = 1/(1-a),$$

the wage elasticity of employment is also constant,

$$-R'/LR'' = 1/(1-b)$$

and both are greater than unity.

Now

$$\Pi = ((1-b)/b)L^b$$

and so Π^*, the maximum of Π along the labour demand schedule $\omega = L^{b-1}$ is

$$\Pi^* = \omega^{*-d}/d,$$

where $d = b/(1-b)$.

This occurs when $\omega = \omega^*$. Similarly

$$U = (L^{b-1} - L^{*b-1})^a L/a$$

so

$$U^* = a^{a-1}(1-b)^a(1-a+ab)^e\omega^{*-e}$$

where

$$e = (1-a+ab)/(1-b)$$

and this occurs when the elasticity of u w.r.t ω is $1/(1-b)$, that is, when $\omega = \omega^*/(1-a+ab)$. Hence we may parameterize Π by L as follows:

$$X = [(L/a)(L^{b-1} - \omega^*)^a]/U^*$$

and

$$Y = [L^b((1-b)/b)]/\Pi^*.$$

For $0 < L \leqslant L^*$ this describes a curve similar to a parabola in (X, Y) space, like that in Figure 6(b). So $\max_{X,Y} \min(X, Y)$ occurs at the optimal intersection of this curve with the line $Y = X$. Thus, the condition for the Kalai–Smorodinsky solution is

$$L^{b-1} = C(L^{b-1} - \omega^*)^a$$

where

$$C = [a(1-b)]^{-a}(1-a+ab)^{-e}\omega^{*1-a}$$

Putting $\alpha = C^{1/a}$ and substituting $L^{b-1} = \omega$, we see that points of intersection are given by any real roots of

$$\omega^{1/a} - \alpha\omega + \alpha\omega^* = 0$$

which are greater than ω^*. Consider the function

$$f(x) = x^{1/a} - \alpha x + \alpha\omega^*.$$

Evidently $f(\omega^*) = \omega^{*1/a} > 0$ and $f(x) > 0$ for large x. Now f attains its minimum at $\xi = a\alpha^{a/(1-a)}$ so an intersection exists provided $f(\xi) \leqslant 0$. Tedious calculation shows that $f(\xi) \leqslant 0$ if and only if

$$a^a(1-b)^a(1-a+ab)^e < (a - a^{1/a})^{1-a}.$$

Consider the case $a = \frac{1}{2}$ and $b = \frac{1}{3}$. Then $d = \frac{1}{2}, e = 1$ and so

$$\Pi^* = 2\omega^{*-1/2} \quad \text{when } \omega = \omega^*$$

and

$$U^* = (4/\sqrt{27})\omega^{*-1} \quad \text{when } \omega = 3\omega^*/2.$$

Now parameterizing by ω we have

$$X = \sqrt{27}\,\omega^*\omega^{-3/2}(\omega - \omega^*)^{1/2}/2$$

and

$$Y = (\omega^*/\omega)^{1/2}$$

so

$$\frac{\mathrm{d}Y}{\mathrm{d}X} = \frac{2\omega(\omega-\omega^*)^{1/2}}{\sqrt{27}\,\omega^{*3/2}(2\omega - 3\omega^*)}.$$

Now $\mathrm{d}Y/\mathrm{d}X = 0$ when $\omega = \omega^*$ (since $\omega \neq 0$ along the labour demand curve), $\mathrm{d}Y/\mathrm{d}X > 0$ when $\omega > 3\omega^*/2$ and $\mathrm{d}Y/\mathrm{d}X < 0$ when $\omega < 3\omega^*/2$. Also $(X, Y) = (0, 1)$ at $\omega = \omega^*$ and $(X, Y) = (1, \sqrt{2/3})$ at $\omega = 3\omega^*/2$. So the image of the labour demand curve $\omega = L^{-2/3}$ in X, Y space is shown in Figure 6(b).

In Figure 6(a) we have shown the points of intersection with the curve $Y = X$ at ω_1 and ω_2. These correspond to the roots of the quadratic equation

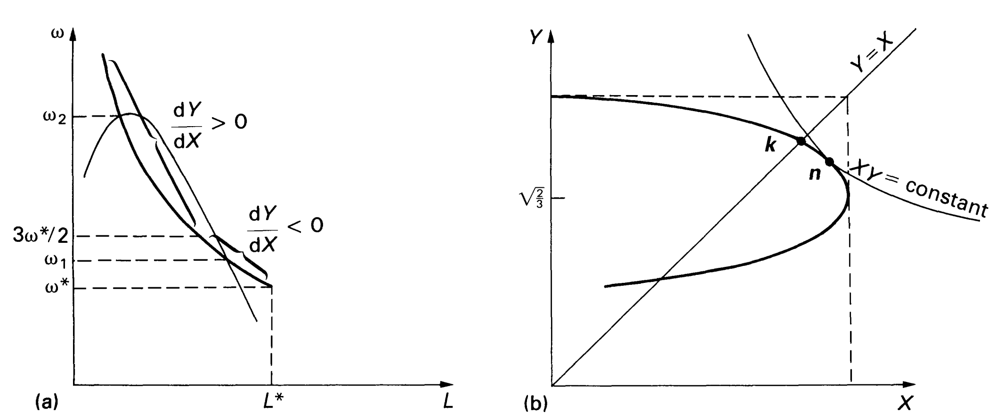

FIG. 6. *Constraint curves.*

$$\omega^2 - \alpha\omega + \alpha\omega^* = 0$$

where $\alpha = 27\omega^*/4$. It is the smaller of these, ω_1, which is the Kalai–Smorodinsky solution and

$$\omega_1 = \frac{\omega^*}{8}(27 - \sqrt{297}).$$

This should be compared with the (symmetric) Nash solution, which occurs when the slope of the constraint curve is $-Y/X$. A short calculation yields

$$\frac{(\omega - \omega)^*}{\omega^*(2\omega - 3\omega^*)} = -1$$

so

$$\omega = \frac{\omega^*(1 + 3\omega^*)}{(1 + 2\omega^*)}.$$

Now the Nash solution assigns a greater wage than the Kalai–Smorodinsky in this example, if

$$\omega^* > (19 - \sqrt{297})/(2\sqrt{297} - 30) = 0.39538.$$

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