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PRICING AND HEDGING CONVERTIBLE BONDS: DELAYED CALLS AND UNCERTAIN VOLATILITY

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Arbitrage-free price bounds for convertible bonds are obtained assuming equity-linked hazard rates, stochastic interest rates and different assumptions about default and recovery behavior. Uncertainty in volatility is modeled using a stochastic volatility process for the common stock that lies within a band but makes few other assumptions about volatility dynamics. A non-linear multi-factor reduced-form equity-linked default model leads to a set of non-linear partial differential complementarity equations that are governed by the volatility path. Empirical results focus on call notice period effects. Increasingly pessimistic values for the issuer's substitution asset obtain as we introduce more uncertainty during the notice period. Uncertain in volatility, in particular, appears to be an important determinant of the call premium that is so often observed in issuer's call policies.

Keywords: Call notice period; call premium; convertible bond; delayed calls; equity-linked default; stochastic interest rates; volatility uncertainty.

1. Introduction

A convertible bond (CB) is a hybrid derivative instrument with complex features that make its value highly sensitive to several risk factors. They are convertible into shares at the investor's decision; the optimality of this decision depends on the equity price, the future spot interest rate, and the probability that the issuer will default. Thus prices are sensitive to stock price and interest rate dynamics, the stock price behavior upon default and to assumptions about recovery and default intensity

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(hazard) rates. Of all these (highly uncertain) quantities the stock price dynamics and default behavior are perhaps the most important. Indeed, in a detailed comparative study of the main types of CB pricing models, Grimwood and Hodges [26] conclude that "accurately modeling the equity process ... appears crucial whereas the intensity rate and spot interest rate processes are of second order importance".

It is an established fact that an equity process with constant volatility, such as that assumed by Black and Scholes [12] is inappropriate for pricing most options. In a large and growing literature (surveyed in [44]) many alternative volatility models have been developed. Volatility models for pricing relatively short-term options, for instance with up to two years to expiry, usually focus on the stochastic dynamics of a variance or volatility process that is correlated with the underlying. Since their short-term dynamics are particularly important, e.g., for pricing exotic, path dependent structures and for hedging any type of option, these models are normally calibrated to be consistent with the current implied volatility surface. But the short-term volatility dynamics are relatively unimportant for pricing CBs. In the primary market where the conversion option is long-dated, CB prices will depend on the realized variance of the stock price over a very long period. Hence it is the *long-term* equity volatility that is the major factor in the accurate pricing of CBs. Unlike equity options, the long-term volatility is the important variable — however, this is extremely difficult to forecast. This problem has been addressed in a new class of "uncertain volatility" models, introduced by Avellaneda *et al.* [6], Avellaneda and Parás [7] and Lyons [38]. These assume only that the volatility process take values within a given interval, with increasing uncertainty in volatility being captured by widening the interval.

Long-term equity volatility also affects the CB price through its specific call features. Most CBs can be called for redemption by the issuer at the effective call price (the clean call price, which is fixed in the covenant, plus accrued interest). Calls are always accompanied by a notice period, of between 15 days and several months, during which the bondholder can elect to convert at any time. When a CB is called, the bondholder has a finite time period during which to convert or to redeem the bond at the effective call price. The call part of the contract is a crucial determinant of the CB value. Call features are attractive to issuers for several reasons: they effectively cap investors profits from a rise in share price and they lessen the uncertainty about issuers' future liabilities because they are a means to force conversion, especially when the firm can refinance at a cheaper rate. On the other hand, call features make the CB less attractive to prospective investors. For this reason the contract usually specifies that the bond is not callable for the first few years after issue (the "hard call protection" period) and following this there may also be "soft-call protection" period, after which the CB is unconditionally callable. During a soft-call protection period the CB is callable only when certain additional conditions are met. For instance, during the soft-call period calls may only allowed when the stock price is sufficiently high, in which case the contract can

specify a "stock price trigger" (e.g., that the common share price exceeds its at-issue price by 50% for a certain number of days). Alternatively, "make-whole" provisions may be made during the soft call protection period. Make-wholes are active during the soft-call period and include premium make-whole and the coupon make-whole features. These additional conditions are designed to increase the attraction of the CB to investors.

The complex features of CBs are difficult to value, especially under uncertainties about the evolution of risk factors. Nevertheless CBs continue to be a very popular asset class. According to Morgan Stanley's ConvertBond.com database, the CB market was worth more than US$500 billion in 2004Q1, with nearly 400 issues between $125 million and $500 million and almost 100 issues in excess of $500 million. According to Grimwood and Hodges [26], the modal CB contract in the ISMA database for the US has a maturity of 15.0 years, pays a 6% semi-annual coupon, and is hard-callable for the first time within three years. Of these contracts, 72% of them have a hard no-call period, and 53% have a put clause. Of the Japanese CBs in the database, 88% have a hard no-call period, 91% have a soft no-call period, 23% have a put clause, while 78% are cross-currency and 56% had a conversion rate re-fix clause.

The literature on valuation models for CBs is very large. Early work was based on the firm value contingent claim approach of Brennan and Schwartz [13, 15] and included the derivation of a closed-form pricing formula for callable convertible bonds with simple features, based on simple assumptions about the evolution of risk factors [32]. The firm value approach has the great advantage that default is endogenous. On the other hand such models become tractable only when the complex structure of firm value is assumed away. For instance, whilst the firm value will be re-estimated each time the CB is marked to market, for the purpose of the model it is normally assumed constant. Consequently most of the recent literature has focused on reduced form equity default models. McConnell and Schwartz [40] noted that modeling the equity price rather than the firm value as a diffusion precluded the possibility of default unless the discount rate is adjusted to account for the possibility of default. This observation has inspired several such "blended discount" approaches. In particular, Derman [22] considered a stock price binomial tree where the discount rate in each time-step is a weighted average of the risky rate and the risk-free (or stock loan) rate, with weight determined by the probability of conversion. In this framework the default event is not explicitly modeled. However compensation for credit risk is included through this "credit-adjusted discount rate".[1] One of the most important papers, by Tsiveriotis and Fernandes [46] provided a rigorous treatment of Derman's ideas by splitting the CB value into equity and bond components, each discounted at respective rates. This approach

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[1] At the same time, Ho and Pfeffer [28] worked with a single stochastic discount factor plus a (constant) credit spread, but this proved difficult because the CB price will be unnecessarily depressed when the equity price is high and the default risk is correspondingly low.

has since been extended to include interest rate and foreign exchange risk factors by Yiğitbaştoğlu [48] and by Landskroner and Raviv [36, 37] who applied a blended discount model to price domestic and cross-currency inflation-linked CBs and to imply credit spreads when the issuing firm has no straight debt and only convertible bonds outstanding.

None of these papers deal explicitly with default. In fact, the implicit assumption is that the stock price does not fall upon the bankruptcy announcement. However in the credit risk literature, [23, 34, 39] many others, default risk is modeled by allowing the stock price to jump downwards at the time of default. Thus the most recent reduced form CB models include a stock price jump on default. Davis and Lischka [21], Takahashi *et al.* [45], Ayache *et al.* [8], Bermudez and Webber [10], Andersen and Buffum [3] and others include jumps in the stock price given default. Most of these models incorporate "equity-linked" hazard rates that are driven by the stock price diffusion and calibrated to the initial term structure of interest rates (e.g., via the Hull and White [30] model).[2] Of particular relevance to this paper is the model introduced by Ayache *et al.* [8]. They propose a single factor model that splits the CB into equity and bond components (as in [46]) and further allows the stock price to jump on default. Their model also permits flexible recovery specifications while explicitly dealing with the default event.

Some of the most interesting research on CBs seeks to explain the issuer's call policy after the soft-call protection period. During the notice period the issuer effectively gives the investor a put on the common share. Thus the optimal call price (the price attained by the common share so that it is optimal for the issuer to call) should be such that the conversion price is *just* greater than the effective call price plus the premium on the put. However, issuers often wait until the conversion price is *significantly* higher than this put before issuing the call. Many reasons have been proposed for this "delayed call" phenomenon. These include the price uncertainty during the call notice period and the issuer's aversion to a sharp stock price decline [1, 4, 15, 25, 33]; the preferential tax treatment of coupons over dividends as an incentive to keep the convertible bonds alive [4, 5, 16, 18] signaling effects, whereby convertible bonds calls convey adverse information to shareholders that management expects the share price to fall [27, 41]; and issuers preferring to let sleeping investors lie [18, 24].

Ingersoll [33] emphasizes the importance of a precise treatment of the firm's behavior with regard to exercising its right to call. When call notice periods are included in the CB valuation model, long-term volatility uncertainty has an additional role to play through its effect on the stock price during the call notice period. However most, though not all, CB models assume volatility is constant. An exception is Andersen and Buffum [3] who assume a deterministic local volatility process. Recently, call notice periods have been included in CB valuation by

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[2]Bermudez and Webber [10] and Barone-Adesi *et al.* [9] also use Hull and White [30] for the short-rate dynamics.

Hoogland *et al.* [29], Butler [15], Lau and Kwok [35] and Grau *et al.* [25]. However all these models assume constant volatility.

This paper first extends the single-factor call notice period model of Grau *et al.* [25] to multiple sources of risk. We investigate the effect of stochastic interest rates on the issuer's optimal call policy when volatility is constant. We then incorporate the uncertain volatility model of Avellaneda *et al.* [6], Avellaneda and Parás [7] and Lyons [38] and find that this is much the most important determinant of the CB value, as suggested by Grimwood and Hodges [26]. Even without special call features CB prices are found to be very sensitive to uncertainty on the long-term volatility of the stock price. When special call features are added, we find that volatility uncertainty provides an intuitive reason for the call premiums of CBs, and the "delayed call" phenomenon for unconditional call with notice period, in particular.

The outline of the paper is as follows. Section 2 describes the valuation framework. We employ a multi-factor reduced-form default model to show how complex call notice and default features can be priced. Following Wilmott *et al.* [47], we derive the linear complementarity problem that captures the CB features in each case. We then address the long-term volatility sensitivity of CBs by assuming the forward volatility is a mean-reverting process. We do not model the volatility diffusion explicitly. Instead we simply assume that volatility remains within certain upper and lower bounds. Then arbitrage-free price bounds for CBs with volatility uncertainty are derived, using the approach introduced by Avellaneda *et al.* [6], Avellaneda and Parás [7] and Lyons [38]. Section 3 presents some useful preliminary results. The price of a CB is a highly complex function of many uncertain risk factors, especially with so many issues having special features nowadays. We therefore illustrate the varying effects of interest rate uncertainty, default behavior and recovery assumptions in the presence of different CB features, before moving on to the main results of this paper. In Sec. 4, examples illustrate the arbitrage-free price and hedge ratio bounds that are derived from an uncertain volatility assumption. These apply to any CB and for many different behavioral assumptions. Our examples include call and put features, interest rate uncertainty, realistic default behavior and appropriate recovery assumptions. A pessimistic approach to pricing the issuer's substitution asset, which comes into existence when the issuer calls the CB, is achieved by widening the volatility uncertainty band. We find that even moderately wider bands during the notice period will capture the call premium. Section 5 concludes.

2. The Valuation Framework

Ayache *et al.* [8] derive a single-factor model with default where the stock price jumps if default occurs. Moreover, the hazard rate is negatively correlated with the stock price. Recovery is also modeled in a flexible way: if the issuer defaults the bondholder has a choice of converting into defaulted common shares or taking

a recovery amount based on either the bond face value or a proportion of the pre-default bond portion value. Grau *et al.* [25] extend this model to include call notice periods. The purpose of this section is to extend these models further: first we include stochastic interest rates (and exchange rate risk if the CB has cross-currency features); secondly we capture the long-term volatility sensitivity of the CB price by assuming the forward volatility process lies within a finite range. Upper bound CB prices are obtained when the volatility realizes its "worst" path in the range. If the trader sells the bond on or above this price he is insured against volatility movements within the range. Such price bounds were shown to be arbitrage-free by Avellaneda and Parás [7].

The CB maturity value is the face value F (plus accrued interest) or the conversion value $\kappa S(T)$, whichever is the greater. The callable and/or putable CB price is bounded above by the maximum of the conversion value and the call price (plus accrued interest) and it is bounded below by the maximum of the conversion value or the put price (plus accrued interest). Accrued interest is defined in the usual way

$$AI(t) = Cpn(t_{n+1})\frac{t - t_n}{t_{n+1} - t_n} \quad \forall\, t \in [t_n, t_{n+1}],$$

where $t_{n+1} - t_n$ is the number of calendar days (given an appropriately chosen day-count convention) and $Cpn[t_{n+1}]$ is the coupon amount paid on the $(n+1)$th coupon date.

2.1. *A two-factor model with reduced-form default*

International or domestic defaultable CBs with no notice period call features are valued first, under a constant volatility assumption. Following Ayache *et al.* [8] and Grau *et al.* [25], we make the simplifying assumption that default risk is diversifiable; hence default probabilities in the "real world" measure and under the equivalent martingale measure are identical. This assumption will, of course, not hold in most practical cases, and parameters of the pricing equations will need to be risk-adjusted. The stock price volatility ν is assumed constant. The dividend-paying rebased stock price is denoted $\tilde{S}(t) = Q(t)S(t)$ where $Q(t)$ is the foreign exchange rate process with volatility ξ.[3] The short rate $r(t)$ follows a correlated mean-reverting [19] diffusion, so that the two risk factors have the dynamics:

$$d\tilde{S}_t(t) = \tilde{S}(t)(r(t) + p\eta - q)dt + \sigma_s dW_1(t) - \eta\tilde{S}(t)dN(t),$$
$$dr(t) = (a - br(t))dt + \sigma\sqrt{r(t)}dW_2(t),$$
$$dW_1(t)dW_2(t) = \rho_{S,r}dt,$$

where $N(t)$ is a Poisson default process which jumps to unity upon default, and is otherwise equal to zero.

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[3]The domestic value of foreign stock, where we assume the CB is domestic currency denominated but converts into foreign stock.

Assume the single default event occurs as the first jump of a Poisson counting process, and let the probability of default over a time interval Δt (contingent on no prior default) be equal to $p\Delta t$ (where p is the risk-neutral hazard rate).

Upon default the stock price suffers a fractional loss in value in reference to its pre-defaulted price as below:

$$\tilde{S}^+ = \tilde{S}^-(1 - \eta),$$

where $0 \leq \eta \leq 1$.

When default occurs, the holder of a convertible has the option to convert immediately into stock worth $\kappa_s \tilde{S}(1 - \eta)$, or to receive an amount equal to RX. Here, X can be the bond face value, the accreted value of the issue price, the bond portion of the convertible "package" just prior default, etc. All such recovery prescriptions are accommodated in the framework. R is the recovery proportion $0 \leq R \leq 1$.

We set up the hedge portfolio. Conceptually, the hedge portfolio, denoted $\Pi(t)$, behaves as follows:

$$\Pi(t) = (1 - \text{pdt}) \times \{\text{undefaulted } P^*\text{-dynamics of portfolio components}\}$$
$$+ \text{pdt} \times \{\text{drop in } \Pi(t) \text{ from default} = \Pi(t^+) - \Pi(t^-)\}.$$

Assume temporarily that the probability of default is zero. We form a hedge portfolio Π consisting of one convertible, α units of zero-coupon treasury bonds of equal maturity (or maturity equal to first call date if the convertible is callable), and β units of the stock (α and β are clearly negative but ignore the signs for now). The zero-coupon treasury is known to follow the SDE (under the risk neutral EMM) given by the well-known Cox, Ingersoll and Ross [19] formula:

$$dB(t,T) = r(t)B(t,T)dt - B(t,T)\tilde{B}(t,T)\sigma_r\sqrt{r}dW_2(t),$$

where

$$\tilde{B}(t,T) = \frac{2(\exp\{(T-t)\sqrt{b^2 + 2\sigma^2})\} - 1)}{2\sqrt{b^2 + 2\sigma^2} + (b + \sqrt{b^2 + 2\sigma^2})(\exp\{(T-t)\sqrt{b^2 + 2\sigma^2}\} - 1)}.$$

Then $\Pi_t = V_t + \alpha B(t,T) + \beta\tilde{S}(t)$ and

$$d\Pi_t = dV_t + \alpha dB(t,T) + \beta d\tilde{S}(t) = V_t dt + V_S d\tilde{S} + V_r dr + \frac{1}{2}V_{SS}d\tilde{S}^2 + V_{Sr}d\tilde{S}dr$$

$$+ \frac{1}{2}V_{rr}dr^2 + \alpha(r(t)B(t,T)dt - B(t,T)\tilde{B}(t,T)\sigma_r\sqrt{r}dW_2(t))$$

$$+ \beta(\tilde{S}(t)(r(t) + p\eta - q)dt + \sigma_s dW_1(t)) + \beta q\tilde{S}(t)dt,$$

recalling that $dN = 0$ on the non-defaulted path. The last term arises from the continuous dividend receivable from holding stock.

The last expression implies:

$$dΠ_t = \left\{ V_t + (r + pη - q)V_S + \frac{1}{2}σ_S^2 \tilde{S}^2 V_{SS} + (a - br)V_r \right.$$

$$+ \left. \frac{1}{2}σ_r^2 r V_{rr} + ρ_{S,r}σ_S σ_r \tilde{S}\sqrt{r}V_{Sr} + αrB + β(r + pη - q)\tilde{S} + βq\tilde{S} \right\} dt$$

$$+ \{σ_S \tilde{S}V_S + βσ_S \tilde{S}\}dW_1(t) + \{σ_r \sqrt{r}V_r - αB\tilde{B}σ_r \sqrt{r}\}dW_2(t),$$

to locally eliminate risk, we require that

$$β = -V_S,$$
$$α = V_r/B\tilde{B},$$

substituting above and after some algebra this gives that on the undefaulted path

$$dΠ_t = \left\{ V_t - q\tilde{S}V_S + \frac{1}{2}σ_S^2 \tilde{S}^2 V_{SS} + (a - br)V_r \right.$$

$$+ \left. \frac{1}{2}σ_r^2 r V_{rr} + ρ_{S,r}σ_S σ_r \tilde{S}\sqrt{r}V_{Sr} + (r/\tilde{B})V_r \right\} dt$$

$$= [V_t + \mathbf{L}V]dt. \tag{2.1}$$

Consider the portfolio value $Π(t)$ contingent upon default having occurred. Then the constituents in the hedge portfolio would behave as follows:

$$Π(t^-) = V(t^-) + αB(t^-, T) + β\tilde{S}(t^-),$$
$$V(t^+) → \text{Max}(κ\tilde{S}(1 - η), RX) \quad \text{(no convertible exists anymore)},^4$$
$$β\tilde{S}(t^+) → β(1 - η)\tilde{S}(t^+) \quad \text{(stock price jumps)},$$
$$αB(t^+, T) → αB(t^+, T) \quad \text{(riskless bond component stays the same)}.$$

Thus the drop in $Π(t)$ at default (excluding the option to convert/take recovery) is:

$$Π(t^-) - Π(t^+) = (αB(t, T) + β(1 - η)\tilde{S}(t) + \text{Max}(κ\tilde{S}(1 - η), RX))$$

$$- V(t) + αB(t, T) + β\tilde{S}(t)$$

$$= -V - ηβ\tilde{S}(t) + \text{Max}(κ\tilde{S}(1 - η), RX). \tag{2.2}$$

The portfolio dynamics can be written as

$$dΠ(t) = (1 - \text{pdt})\{\text{no default incremental change in value of } Π(t)\}$$
$$+ (\text{pdt})\{Π(t^-) - Π(t^+)\}.$$

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[4]But the option to convert into defaulted shares or R times some recovery amount X comes into existence. See below.

Adding all terms together gives the portfolio dynamics in the presence of default, with the recovery and stock price jump assumptions given before. Therefore

$$d\Pi(t) = (1 - \text{pdt}) \left\{ V_t - q\tilde{S}V_S + \frac{1}{2}\sigma_S^2\tilde{S}^2 V_{SS} + (a - br)V_r \right.$$

$$\left. + \frac{1}{2}\sigma_r^2 r V_{rr} + \rho_{S,r}\sigma_S\sigma_r\tilde{S}\sqrt{r}V_{Sr} + (r/\tilde{B})V_r \right\} dt$$

$$- \text{pdt}\{V(t) + \eta\beta\tilde{S}(t) - \text{Max}\{\kappa_s\tilde{S}(t)(1 - \eta), RX\}\}.$$

As we have assumed the default risk to be hedged, the portfolio will earn the risk-free rate over the next time increment. Hence

$$d\Pi(t) = r\{V(t) + \alpha B(t,T) + r\beta\tilde{S}(t)\}dt = \{rV(t) + r[V_r/\tilde{B}(t,T)] - r\tilde{S}(t)V_S\}dt.$$

Equating these expressions we get

$$\left\{ V_t + (r - q + p\eta)\tilde{S}(t)V_S + \frac{1}{2}\sigma_S^2\tilde{S}^2 V_{SS} + (a - br)V_r \right.$$

$$\left. + \frac{1}{2}\sigma_r^2 r V_{rr} + \rho_{S,r}\sigma_S\sigma_r\tilde{S}\sqrt{r}V_{Sr} - (r + p)V(t) \right\}$$

$$+ p\,\text{Max}\{\kappa_s\tilde{S}(t)(1 - \eta), RX\} = 0.$$

However the possibility of early exercise with put and call features results in inequality. Spefically it can be that:

$$\left\{ V_t + (r - q + p\eta)\tilde{S}(t)V_S + \frac{1}{2}\sigma_S^2\tilde{S}^2 V_{SS} + (a - br)V_r \right.$$

$$\left. + \frac{1}{2}\sigma_r^2 r V_{rr} + \rho_{S,r}\sigma_S\sigma_r\tilde{S}\sqrt{r}V_{Sr} - (r + p)V(t) \right\}$$

$$+ p\,\text{Max}\{\kappa_s\tilde{S}(t)(1 - \eta), RX\} \gtreqless 0,$$

which using the differential operator $\mathbf{P}$ is re-written as

$$V_t + \mathbf{P}V - (r + p)V(t) + p\,\text{Max}\{\kappa_s\tilde{S}(t)(1 - \eta), RX\} \gtreqless 0. \tag{2.3a}$$

$$\mathbf{P}V = (r - q + p\eta)\tilde{S}(t)V_S + \frac{1}{2}\sigma_S^2\tilde{S}^2 V_{SS} + (a - br)V_r + \frac{1}{2}\sigma_r^2 r V_{rr} + \rho_{S,r}\sigma_S\sigma_r\tilde{S}\sqrt{r}V_{Sr}. \tag{2.3b}$$

The linear complementarity constraints (LCC) describe how $V(t)$ will depend on the effective call price, the effective put price and the conversion value at every point in the grid. Specifically, two cases are distinguished:

LCC 1:

(i) If $B_c(t) + AI(t) \leq \kappa S(t)$ then $V(t) = \text{Max}\{\kappa S(t), B_c(t) + AI(t)\} = \kappa S(t)$.

That is, if at any point in the grid the effective call price $B_c(t) + AI(t)$ is less than the conversion value $\kappa S(t)$, the issuer calls the bond for redemption. Since the bondholder may still convert, he has the choice of taking the effective call price or conversion value, the latter being greater.

$$\text{(ii) If } B_c(t) + AI(t) > \kappa S(t) \quad \text{then}$$

either:

$$V_t + \mathbf{P}V(t) - (r+p)V(t) + p \, \text{Max}\{\kappa S(t)(1-\eta), RX\} \leq 0$$
$$\&$$
$$V(t) = \text{Max}\{\kappa S(t), B_p(t) + AI(t)\},$$

that is, if the continuation value of the CB falls below its conversion or effective put value it is optimal for the bondholder to convert;
or

$$V_t + \mathbf{P}V(t) - (r+p)V(t) + p \, \text{Max}\{\kappa S(t)(1-\eta), RX\} \geq 0$$
$$\&$$
$$V(t) = B_c(t) + AI(t),$$

that is, it is optimal for the issuer to terminate the time value of the uncalled CB and call it for redemption;
or

$$V_t + \mathbf{P}V(t) - (r+p)V(t) + p \, \text{Max}\{\kappa S(t)(1-\eta), RX\} = 0$$
$$\&$$
$$\text{Max}\left\{\kappa S(t), B_p(t) + AI(t)\right\} \leq V(t) \leq B_c(t) + AI(t),$$

that is, it is neither optimal for the issuer to call (and pay more than the uncalled value of the CB), nor for the bondholder to convert, as the continuation value of the CB is higher.

Note that the specification of the linear complementarity conditions remains unchanged when the hazard rate is assumed to depend on the stock price, i.e., $p = p(S)$. The hazard rate dynamics affect $V(t)$ through the replicating portfolio dynamics (2.3a), and it is easy to accommodate a stochastic (inversely correlated) functional form such as in Ayache *et al.* [8], Grau *et al.* [25] and Andersen and Buffum [3]. We assume

$$p(S) = p(0)(S(t)/S(0))^{\alpha}.$$

Such a form was tested on Japanese corporate bonds by Muromachi [42] and $-2 \leq \alpha \leq -1.2$ was found to provide an adequate fit for that market.

2.2. *Uncertain volatility*

Volatility diffusions increase the dimension of the pricing SDE. Thus, for multi-factor derivatives such as CBs, the resolution of the pricing SDE is computationally

difficult, even using finite difference methods. The implementation problems of these models have already discussed by many authors (for instance, see [2, 6]). But, at the opposite end of the spectrum, the constant volatility assumption is very unrealistic for long-dated contracts such as CBs and could lead to large errors if applied in practice. Since long-term implied volatilities are either unreliable or not available, traders might simply price and hedge CBs using a point volatility estimate, and adjust estimates only for risk premia and transaction costs. But many traders would also consider a finite range or "band" that, in their view, is likely to bound the volatility. Taking the highest price over that range will then provide a lower bound for their profit margin. This concept has been formalized in the uncertain volatility models of Avellaneda and Parás [7], Lyons [38], and Avellaneda *et al.* [6].

We now incorporate these practical features of CB pricing in the PDE framework. The assumption of randomness in stock price volatility is maintained, but beyond this we make no parametric assumptions regarding its dynamics, and assume only that it lies in some band, $\sigma_S(t) \in [\sigma_{S,L}(t), \sigma_{S,H}(t)]$. The band may be imputed from the quantiles of the historical volatility distribution, from implied volatilities, from a Bayesian prior or from any combination of these. For reasons of space, we do not delve into the statistical methods that can be used to compute these bands. We remark only that, for instance, the historical volatility distribution could be based on the in-sample volatility term structure estimates of a suitable GARCH process applied to a long history of daily returns on the common shares of the CB issuer. The mean reverting behavior of volatility implies that the variability of volatility estimates decreases as the volatility maturity increases. Thus the empirical distribution of volatility will typically result in a narrowing volatility range as maturity increases.

The arbitrage-free CB price bounds depend on the worst and best realization of volatility within its band. In the framework developed by Avellaneda and Parás [7], Lyons [38], and Avellaneda *et al.* [6] valuation becomes a stochastic control problem where the volatility term in the pricing equation switches, depending on the gamma of the instrument. The numerical convergence properties of such models are discussed in Pooley *et al.* [43].

Just as in Sec. 2.1, the construction of a hedge portfolio leads to the CB valuation equation. Note that (2.3b) contains the following terms in stock price volatility that are influenced by its uncertainty:

$$\mathbf{L}V(t) = \frac{1}{2}\sigma_S^2 S^2 V_{SS} + \rho_{S,r}\sigma_S\sigma_r S\sqrt{r}V_{Sr}. \tag{2.4}$$

Because we are uncertain about the path of $\sigma_S(t)$ in a long-term band, we price the CB pessimistically to obtain the "best" price and optimistically to obtain the "worst" price. Consider first the "best" price. The task of the numerical algorithm is to switch the stock price volatility to $\sigma_{S,H}(t)$ at every discretization point in the grid whenever

$$\frac{1}{2}\sigma_{S,H}^2 S^2 V_{SS} + \rho_{S,r}\sigma_{S,H}\sigma_r S\sqrt{r}V_{Sr} \geq \frac{1}{2}\sigma_{S,L}^2 S^2 V_{SS} + \rho_{S,r}\sigma_{S,L}\sigma_r S\sqrt{r}V_{Sr}, \tag{2.5a}$$

and to switch to $\sigma_{S,L}(t)$ at every discretization point in the FD grid whenever

$$\frac{1}{2}\sigma_{S,H}^2 S^2 V_{SS} + \rho_{S,r}\sigma_{S,H}\sigma_r S\sqrt{r}V_{Sr} < \frac{1}{2}\sigma_{S,L}^2 S^2 V_{SS} + \rho_{S,r}\sigma_{S,L}\sigma_r S\sqrt{r}V_{Sr}. \quad (2.5b)$$

For the "worst" price the numerical algorithm now switches the volatility to $\sigma_{S,L}(t)$ at every discretization point in the grid when

$$\frac{1}{2}\sigma_{S,H}^2 S^2 V_{SS} + \rho_{S,r}\sigma_{S,H}\sigma_r S\sqrt{r}V_{Sr} \geq \frac{1}{2}\sigma_{S,L}^2 S^2 V_{SS} + \rho_{S,r}\sigma_{S,L}\sigma_r S\sqrt{r}V_{Sr}, \quad (2.6a)$$

and switches to $\sigma_{S,H}(t)$ at every discretization point on the grid when

$$\frac{1}{2}\sigma_{S,H}^2 S^2 V_{SS} + \rho_{S,r}\sigma_{S,H}\sigma_r S\sqrt{r}V_{Sr} < \frac{1}{2}\sigma_{S,L}^2 S^2 V_{SS} + \rho_{S,r}\sigma_{S,L}\sigma_r S\sqrt{r}V_{Sr}. \quad (2.6b)$$

The switching volatility regime results in a non-linear parabolic PDE in two-dimensions. This is then re-formulated as a linear complementarity problem that captures the conversion, put, and call features of the CB.

2.3. *Call notice periods*

Now we allow that, if the issuer calls the CB at time t, a notice period of length τ applies during which the bondholder can elect to convert or wait until $t + \tau$ to redeem the bond for the effective call price at that time. On calling the CB the issuer effectively delivers a "substitution asset" consisting of common shares plus a European style put with maturity $t + \tau$ and strike equal to the dirty call price. We denote the value of this substitution asset by $V_C(t)$. Inclusion of the notice period will increase the numerical burden considerably, since the upper constraint in the PDE is itself the solution of another "sub-PDE" to be satisfied by the substitution asset. Notice periods are typically between fifteen days and several months duration. Hence the sub-PDE refers to an asset with significantly smaller maturity than the CB.[5]

The modified payoff at redemption becomes $\mathrm{Max}\{\kappa S(t+\tau), CP(t)+AI(t+\tau)\}$. This resembles the payoff of a CB with a reduced time value but with much higher face value. As the conversion option is American and put provisions still apply in $(t, t+\tau)$ the substitution asset value has no closed form solution (unless no coupons or dividends are payable in the notice period, in which case a closed-form solution does exist). The substitution asset must be solved for at each time step of the CB in addition to the normal time-stepping routine for the CB itself. Then this auxiliary asset's value constitutes the upper linear complementarity constraint. When the uncalled CB is more valuable than this, it is in the firm's best interest to call the CB for redemption.

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[5]But whilst fewer time steps are needed for convergence, the computational time still increases linearly in the number of time-steps in the sub-PDE. Evidently the notice period results in computational cost that scales by a factor equal to the number of sub-time-steps for substitution asset.

We formalize this as follows. Recall the valuation equation with default and uncertain volatility but without notice periods is:

$$V_t + \mathbf{PV}(t) - (r + p(S))V(t) + p(S)\,\text{Max}\{\kappa S(t)(1 - \eta), RX\} \overset{>}{\underset{<}{=}} 0, \qquad (2.7\text{a})$$

with

$$\mathbf{PV}(t) = (r - q + p(S)\eta)S(t)V_S + (a - br)V_r + \frac{1}{2}\sigma_r^2 rV_{rr} + \mathbf{LV}(t), \qquad (2.7\text{b})$$

where $\mathbf{LV}(t)$ is defined by (2.4). Furthermore the numerical algorithms determines $\mathbf{LV}(t)$ using "best" price volatilities defined by (2.5a) and (2.5b), and "worst" price volatilities defined by (2.6a) and (2.6b).

The linear complementarity constraints may now be formulated in terms of the value of the substitution asset as follows:

LCC 2:

(i) If $V_C(t) \leq \kappa S(t)$ then $V(t) = \kappa S(t)$.

That is, if the substitution asset is less valuable than the conversion value it is in the issuer's interest to call and terminate the time value of the CB, whereupon the bondholders will elect to convert because the conversion price is more valuable than the substitution asset that replaces the CB.

(ii) If $V_C(t) > \kappa S(t)$ then

either:

$$V_t + \mathbf{PV}(t) - (r + p(S))V(t) + p(S)\,\text{Max}\{\kappa S(t)(1 - \eta), RX\} \leq 0$$
$$\&$$
$$V(t) = \text{Max}\{\kappa S(t), B_p(t) + AI(t)\},$$

that is, the continuation value of the CB falls below its conversion or effective put value and it is optimal to convert;

or

$$V_t + \mathbf{PV}(t) - (r + p(S))V(t) + p(S)\,\text{Max}\{\kappa S(t)(1 - \eta), RX\} \geq 0$$
$$\&$$
$$V(t) = V_C(t),$$

that is, the uncalled CB is more valuable than the substitution asset and it is optimal to call;

or

$$V_t + \mathbf{PV}(t) - (r + p(S))V(t) + p(S)\,\text{Max}\{\kappa S(t)(1 - \eta), RX\} = 0$$
$$\&$$
$$\text{Max}\{\kappa S(t), B_p(t) + AI(t)\} \leq V(t) \leq V_C(t),$$

that is, continuation is optimal because (a) conversion and/or put is less valuable to bondholders and (b) if the issuer called it would deliver a more valuable substitution asset than the current (uncalled) CB value.

Ingersoll [32], Asquith [4], and others have cited the firm's natural aversion to a "busted call" as an important factor explaining the delayed call phenomenon. That is, the issuer fears that after it calls the CB, the stock price falls significantly during $t + \tau$. If this happens the issuer is faced with a "busted call" and the undesirable outcome of having to redeem the bond at the much higher dirty call price in cash. Then the redemption payment can lead to cash flow problems and it may even need to float a new issue to finance the payment.

The uncertain volatility CB valuation framework can be used to formalize the busted call explanation of firms' delayed call policies. Fearing a busted call, the substitution asset should be priced on the assumption that volatility will be particularly high during the notice period. Thus, for instance, the upper bound for volatility can be based on the maximum historical τ-period volatility. To incorporate this feature we include the possibility that the issuer assumes a *wider* band of volatility uncertainty during the notice period.

At each time-step in the grid for $V(t)$ before completing the routine, $V_C(t)$ needs to be computed. The inequality constraint satisfied by $V_C(t)$ is

$$V_{C,t} + \mathbf{P}V_C(t) - (r + p(S))V_C(t) + p(S)\text{Max}\{\kappa S(t)(1 - \eta), RX\} \leq 0, \qquad (2.8)$$

where now the stock price volatility in $\mathbf{P}V_C(t)$ switches according to between wider extremes. In particular, we assume that $\sigma_{S,H}(t)$ increases during the call notice period.

The linear complementarity constraints become:

LCC 3:

$$V_{C,t} + \mathbf{P}V_C(t) - (r + p(S))V_C(t) + p(S)\text{Max}\{\kappa S(t)(1 - \eta), RX\} < 0$$
$$\&$$
$$V_C(t) = \text{Max}\{\kappa S(t), B_p(t) + AI(t)\},$$

or

$$V_{C,t} + \mathbf{P}V_C(t) - (r + p(S))V_C(t) + p(S)\text{Max}\{\kappa S(t)(1 - \eta), RX\} = 0$$
$$\&$$
$$V_C(t) \geq \text{Max}\{\kappa S(t), B_p(t) + AI(t)\}.$$

2.4. *Recovery*

As in Ayache *et al.* [8], a range of different recovery assumptions can be modeled. Recovery is captured by the term RX in the pricing equation (2.7a) with $0 \leq R \leq 1$ and X being a proportion of either face value or the market value of the bond component. The recovery can be any portion of face value or of the market value

of the bond portion of the CB prior to default. If face is recovered, the PDE is totally uncoupled: we simply replace X by F and solve the resulting single PDE. The recovery of a portion of the face value certainly makes the problem easier to solve, but it may be more appropriate to assume bondholders will be left with a proportion of the pre-default market value when the issuer goes bankrupt. This is more involved. We follow Ayache *et al.* [8], who suggest an effective splitting of the CB into bond and equity components in the spirit of Tsiveriotis and Fernandes [46].

2.5. *Discretization*

PDEs are solved using the Crank and Nicholson [20] scheme with successive over-relaxation (SOR). For the bond-portion recovery model, we discretize the model as follows. Denote by $\tilde{B}^n$ the post-iterations value of B at the nth calendar time-step, prior to application of the LCC2 and LCC3 constraints and denote by B^{n+1} the known $n+1$th time-step value of B. We obtain $\tilde{B}^n$ using the SOR iterative method on the discretized bond component PDE. Time-stepping using a θ-advancement scheme (and we use Crank-Nicholson, with $\theta = 1/2$) is implemented by writing the θ-averaged system of simultaneous equations in B^{n+1} and $\tilde{B}^n$:

$$-\frac{B^{n+1} - \tilde{B}^n}{\Delta t} = \theta(\mathbf{P}B)^{n+1} + (1-\theta)(\mathbf{P}\tilde{B})^n - \theta(r + p(S))B^{n+1}$$
$$- (1-\theta)(r + p(S))\tilde{B}^n + \theta p(S)RB^{n+1} + (1-\theta)p(S)R\tilde{B}^n,$$

which implies

$$\left(\frac{1}{\Delta t}\mathbf{1}_{(M+1)(L+1)} - (1-\theta)\mathbf{P}^n\right)\tilde{B}^n + (1-\theta)(r + (1-R)p(S))\tilde{B}^n$$
$$= \left(\frac{1}{\Delta t}\mathbf{1}_{(M+1)(L+1)} + \theta\mathbf{P}^{n+1}\right)B^{n+1} - \theta(r + (1-R)p(S))B^{n+1},$$

where $M + 1$ is the number of spatial grid points for the discretized stock price, $L+1$ is the number of spatial grid points for the discretized short term interest rate, and $\mathbf{1}_{(M+1)(L+1)}$ is the block-diagonal $(M + 1)(L + 1)$-dimensional identity matrix.

For the equity component C, denote by $\tilde{C}^n$ the post-iterations value of C at the nth calendar time-step, prior to application of the LCC2 and LCC3 constraints and denote by C^{n+1} the known $n + 1$th time-step value of C. We obtain $\tilde{C}^n$ using the SOR iterative method on the discretized bond component PDE, again time-stepping using the Crank-Nicholson scheme.

This leads to:

$$-\frac{C^{n+1} - \tilde{C}^n}{\Delta t} = \theta(\mathbf{P}C)^{n+1} + (1-\theta)(\mathbf{P}\tilde{C})^n - \theta(r + p)C^{n+1} - (1-\theta)(r + p)\tilde{C}^n$$
$$+ \theta p\,\mathrm{Max}(\kappa S(1 - \eta) - RB^{n+1}, 0)$$
$$+ (1-\theta)p\,\mathrm{Max}(\kappa S(1 - \eta) - R\tilde{B}^n, 0),$$

and notice the coupling with the bond portion above. The above system implies:

$$\left(\frac{1}{\Delta t}\mathbf{1}_{(M+1)(L+1)} - (1-\theta)\mathbf{P}^{n+1}\right)\tilde{C}^n + (1-\theta)(r+p)\tilde{C}^n$$

$$= \left(\frac{1}{\Delta t}\mathbf{1}_{(M+1)(L+1)} + \theta\mathbf{P}^n\right)C^{n+1} - \theta(r+p)C^{n+1}$$

$$+ \theta p\operatorname{Max}(\kappa S(1-\eta) - RB^{n+1}) + (1-\theta)p\operatorname{Max}(\kappa S(1-\eta) - R\tilde{B}^n).$$

The CB price at each time-step is then the sum of B^n and C^n.

2.6. *Terminal and boundary conditions*

The substitution asset matures at $t + \tau$, at which point the holder chooses between redemption and conversion. Thus the terminal condition for the substitution asset is

$$V_C(S, r, t + \tau) = \operatorname{Max}\{\kappa S(t+\tau), B_c(t+\tau) + AI(t+\tau)\}. \tag{2.9}$$

At maturity, by converting the bondholder must pay the strike price $F - AI(T)$, which is similar to a warrant, so the terminal condition for the CB is:

$$V(S, r, T) = F + AI(T) + \operatorname{Max}\{\kappa S(T) - F - AI(T), 0\}$$
$$= \operatorname{Max}\{\kappa S(T), F + AI(T)\}. \tag{2.10}$$

At the upper boundary for S we assume the price is linear in S, as the convertible bonds will be almost certainly converted as $S \to \infty$. Thus $V(S_{\max}) = \kappa S_{\max}$ or more precisely, $V(M\Delta S, r, t) = \kappa M\Delta S$, so $V_S(M\Delta S, r, t) = \kappa$ and $V_{SS}(M\Delta S, r, t) = V_{Sr}(M\Delta S, r, t) = 0$.

   The boundary at $S = 0$ is implicitly defined in the PDE. As the terms in S vanish the PDE becomes identical to that of a corporate bond. Due to the CIR assumption, interest rates are non-negative and the lower spatial boundary for the interest rate is $r = 0$. The PDE at $r = 0$ is approximated to second order accuracy, as several of the spatial terms in r vanish on the domain. Then the PDE (2.7a) reduces to

$$V_t + aV_r - \gamma SV_S + \tfrac{1}{2}\sigma_s^2 S^2 V_{SS} - rV(t) \overset{>}{\underset{<}{=}} 0 \quad \forall\, t \in [0, T].$$

The upper boundary for r is more difficult to specify. The case $r \to \infty$ is easily visualized as in that case the bond floor drops to zero.[6] In practice (due to the mean-reverting CIR specification) we truncate the r domain at a much lower level. We assume $V_r < \infty$ and $V_{sr} < \infty$ on the upper boundary for r.

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[6]For a rigorous treatment regarding the r $\to \infty$ boundary conditions, we refer to Barone-Adesi *et al.* [9].

3. Preliminary Results

In this section we examine the empirical features of the model when volatility is constant. In Sec. 3.1 we price a CB with very simple call and put features and with fixed default and recovery rates to assess the effect of interest rate uncertainty on the CB price. We also examine how price changes with moneyness by changing the conversion ratio κ. Section 3.2 looks at the behavior of the CB price and hedge parameters as we change the call and put features and Sec. 3.3 introduces different recovery assumptions, hazard rates and stock loss rates.

3.1. *Quantifying interest rate risk*

We consider a 10-year semi-annual CB with 7% coupon including a call at year 4 (without notice period) and two put possibilities at years 4 and 7. Such features will primarily affect the bond value of the CB. The stock price diffusion has a volatility of 38% and a risk premium $\gamma = 0.015$. Upon default 20% of face value is recovered, the stock price drops by 30%, while the hazard rate is 0.03. Initially the conversion ratio is unity. We then set $\kappa = 0.5$ to add even more weight to the bond value of the CB. In this and all subsequent examples the face value of the bond is 100\$ and the initial stock price is also 100\$.

Three cases are considered[7]:

(a) A one-factor model where the term structure of rates is flat, at 5%.
(b) A deterministic term structure model, assuming an upward sloping yield curve obeying the CIR equation with the diffusion set to zero. The short-term interest rate is 5%, the long-term interest rate is 7.5% and the mean reversion intensity is high (0.8).
(c) A two-factor model with the same parameters as (b) but with a CIR short rate volatility of 22% and a correlation between S and r of 0.1.

The term sheets for the CB and its price and hedge ratios for the three models are shown in Table 1.[8]

In our example, the steep upward sloping yield curve in model (b) has a marked effect on the price. Increasing the long-term rate decreases the bond value of the CB, which is the dominant effect for this CB. Thus, from the flat yield curve price of 143.87 in model (a), the price decreases to 137.41 — a 4.5% difference (and note

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[7]CB pricing models that assume a flat term structure of interest rates include those of McConnell and Schwartz [40], Derman [22], Tsiveriotis and Fernandes [46], Ayache *et al.* [8] and Grau *et al.* [25]. Several authors allow for deterministic interest rates, including Hoogland *et al.* [29], Takahashi *et al.* [45], Hung and Wang [31], and Andersen and Buffum [3]. Many CB pricing models include stochastic interest rates, including those developed by Carayannopoulos [17], Davis and Lischka [21], Yigitbaşioğlu [48], Barone-Adesi *et al.* [9], Grimwood and Hodges [26] and Bermudez and Webber [10].

[8]In the tables M denotes the number of subdivisions of $[0, S_{max}]$, L denotes the number of subdivisions of $[0, r_{max}]$, N denotes the number of subdivisions of $[0, T]$, and T_{sub_steps} denotes the number of time-steps in the algorithm for pricing the substitution asset.

Table 1. Effect of Stochastic interest rates and conversion premium.

General		Cash Flows	
Maturity	10.0	Face	$100
γ	0.015	Coupon	0.07
σ_s	0.38	Frequency	2
Grid Dimensions		Credit Model	
S_{max}	$1000	p	0.03
r_{max}	0.20	η	0.3
N	800	R	0.2
M	200		
L	20		
Call and Notice Periods			
Time	Price	Notice	N/A
4.0	155.0	Period	

	CIR	Put Features		$x = 1.0$		$x = 0.5$	
				(a)			
a	0	Time	Price	**Price**	143.87	**Price**	114.3
b	0	4.0	103.5	ModelΔ	0.6877	ModelΔ	0.235
σ_r	0	7.0	108.5	Model Γ	0.0031	Model Γ	0.00015
ϱ_r	0						
				(b)			
a	0.06	Time	Price	**Price**	137.41	**Price**	106.28
b	0.8	4.0	100.0	ModelΔ	0.7237	ModelΔ	0.2429
σ_r	0	7.0	105.0	Model Γ	0.0003	Model Γ	0.00183
ϱ_{sr}	0						
				(c)			
a	0.06	Time	Price	**Price**	138.63	**Price**	107.44
b	0.8	4.0	100.0	ModelΔ	0.7213	ModelΔ	0.2459
σ_r	0.22	7.0	105.0	Model Γ	0.00293	Model Γ	0.00176
ϱ_{sr}	0.1						

that higher coupons will increase this difference further). At the same time, the delta of the CB with respect to the equity increases. This is because increasing the long-term rate also increases the value of the conversion option.

The addition of interest rate volatility could have little effect on the CB price when there is a positive correlation between interest rates and stock price and the conversion ratio is unity. When a rise in interest rates is likely to be accompanied by an increase in stock price, whilst the bond value decreases the value of the option to convert into equity will increase. Conversely, any stochastic movement in interest rates downwards would increase the bond value but this would be offset by a decrease in the equity value of the CB. Of course, under negative equity-interest rate correlation the price effects of stochastic interest rates would be greater.

Even with the high level of interest rate volatility chosen in our example, the positive equity-interest rate correlation means that the CB price is not much affected. It does increase marginally, to 138.63, a rise of about 0.9%. A decrease in the conversion ratio should cause the interest rate uncertainty effect on the bond value to dominate. With a low conversion ratio ($\kappa = 0.5$), stochastic interest rates produce a rise in price of 1.1%, from 106.28 to 107.44. The main effect of decreasing the conversion ratio is, of course, to decrease the stock delta: from about 0.72 to about 0.24 in our example. Interest rate uncertainty also has a small but noticeable effect on the delta. This increases by 1.2%, from 0.2429 (with deterministic rates) to 0.2459 (with stochastic rates).

In summary, even in the absence of special put and call features, the accurate modeling of interest rates can be important for pricing as well as hedging, particularly when the conversion ratio is low and equity-interest rate correlation is negative. Our examples indicate that the choice between stochastic and deterministic rates can influences prices by more than 1% even when the equity-interest rate correlation is positive. Also, compared with hedge ratios based on stochastic interest rates, the investor could be under-hedged in stock by 1% or more if deterministic rates are assumed, and by 4% or more if the CB is priced using a flat term structure.

3.2. *Call and put features with equity-linked hazard rates*

Calls and puts are increasingly common features of CBs. Grimwood and Hodges [26] report that 72% of US convertibles in the ISMA database are callable and 99% of the Japanese convertibles are callable. Call and put features reduce the lifetime of CBs. When common share volatility is high, the call feature is a valuable option to force conversion. Without notice periods the issuer should call when the CB price exceeds the effective dirty call price at any given time. The issuer will call to curtail the time value of the CB, which is an increasing function of the volatility. The put feature in CBs on the other hand is expected to be especially valuable when the probability of default is high, when the conversion value is low, or when the hazard rate is equity-linked. The following example illustrates the significant effect of these features on CB prices and hedge ratios.

Consider a 5-year CB (convertible into one common share with current price $S(0) = 100$), paying 8% coupon semi-annually. The underlying shares do not pay dividends. Interest rates are first assumed to be flat at $r(0) = 5\%$ while CB prices and deltas are computed to measure the influence of call and put features as equity volatility and hazard rates are varied. Thus $(a, b, \sigma_r, \rho_{s,r}) = (0, 0, 0, 0)$ while $(\eta, R, \alpha) = (0.3, 0, 0)$. We consider three scenarios:

   I: Non-callable and non-putable
  II: Callable in 3 years at price \$170, non-putable
 III: Putable in 3 years at price \$109 and callable in 3 years at price \$140

Results are summarized in Table 2.

Table 2. Effect of call and put features with varying (a) volatility and (b) hazard rate.

	$\sigma_S(p=0.02)$				
	20%	35%	50%	65%	80%
Price ($)					
I	137.1	146.1	156.4	165.2	172.9
II	135.1	143.6	152.0	159.8	166.9
III	133.7	142.6	151.3	159.4	167.0
Delta					
I	0.79	0.774	0.795	0.826	0.854
II	0.717	0.718	0.746	0.78	0.813
III	0.667	0.686	0.720	0.756	0.791
	$p(\sigma_S=20\%)$				
	0	0.01	0.02	0.04	0.08
Price					
I	140.1	138.6	137.1	134.5	130.1
II	138.0	136.5	135.1	132.5	128.1
III	136.0	134.6	133.2	130.6	126.6
Delta					
I	0.756	0.775	0.792	0.823	0.873
II	0.679	0.698	0.717	0.751	0.808
III	0.634	0.651	0.667	0.694	0.749

General comments on these results are:

 (i) The price of a CB increases with volatility, regardless of put or call features.
 (ii) The addition of a put feature increases the CB value, and the addition of a call feature decreases the value. These effects increase as equity volatility increases.
(iii) The value of the protection offered by a put feature also increases with the default probability.
(iv) Increasing probability of default has a uniformly detrimental effect on prices, while monotonically increasing the delta.
 (v) The delta drops sharply with the introduction of both call and put features.
(vi) The reduction in delta through adding a call feature decreases with the hazard rate.

The results in Table 2 were, however, based on unrealistic assumptions about interest rate and default behavior, with a flat, constant yield curve and with hazard rates assumed independent of the equity value. So let us now consider how call and put features affect CB prices within a more detailed CB model, and in the context of a more realistic example. Consider again a 5-year CB with 8% coupon paid semi-annually, but now assume a conversion ratio of 70% and a 30% recovery rate. We also assume the stock price drops by 30% upon default; the holder then has the choice of taking 30% of face value (i.e., $30) or converting into shares. The yield curve is upward sloping and volatile with a long rate of 10%. The equity

price diffusion has $\gamma = 0.024$ and the interest rate diffusion has CIR parameters $(a, b, \sigma_r, \rho_{s,r}) = (0.08, 0.8, 0.275, -0.3)$.

We now consider the price effect of call and put features for different levels of stock price volatility, assuming an equity-linked hazard rate and, using the form suggested by Muromachi [42] with default characteristics $(p(0), \eta, R, \alpha) = (0.04, 0.3, 0.3, -1.2)$. The three scenarios are:

  I. Non-callable and non-putable
 II. Putable in 3 years at price \$109.0
III. Callable in 3 years at price \$140.0

Figure 1 illustrates the price effect of these features for different levels of volatility and in the presence of (a) constant and (b) equity-linked hazard rates. It is seen that the price effects of call and put features are dominated by the assumption regarding hazard rate dynamics, especially when volatility is high. The difference between constant and equity-linked hazard rate assumptions is less pronounced when there is a put feature because the protection offered by the put renders the CB less sensitive to hazard rate dynamics. On the other hand, adding a put feature is even more valuable when hazard rates are equity-linked. Table 3 shows that the put feature gives price increases of between 6% and 10%, increasing with the level of equity volatility. In general, introducing the call feature decreases the price by between 1% and 3% — calls still have more noticeable price effects under high volatility. However, compared with put features the call feature price effects are less sensitive to the hazard rate dynamics.
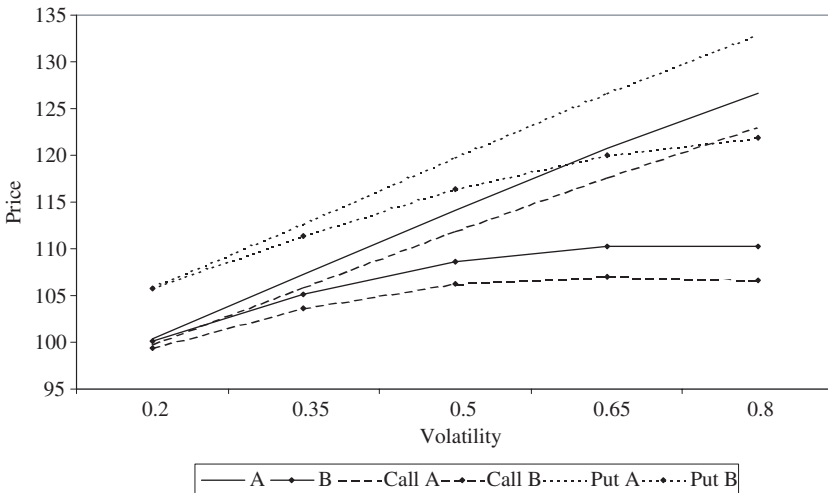

Fig. 1. Price effect of call and put features for different common share volatility with (A) constant and (B) equity-linked hazard rates.

Table 3. Effect of call and put features with equity linked hazard rates and stochastic interest rates.

Price ($)	σ_S				
	20%	35%	50%	65%	80%
Stochastic Interest Rates					
I	100.08	105.07	108.58	110.22	110.23
II	105.78	111.37	116.42	120.04	121.83
III	99.43	103.59	106.20	107.04	106.64
Deterministic Interest Rates					
I	99.71	104.83	108.23	109.72	109.72
II	105.54	111.22	116.20	119.69	121.33
III	99.04	103.37	105.94	106.68	106.22

Equity-linked hazard rates enhance the effect of interest rate uncertainty on the CB price, particularly at high levels of equity volatility. This is because high equity volatility increases the default probability and, with stochastic interest rates, the cash amount recovered becomes more sensitive to interest rates. For instance, re-pricing the putable but non-callable CB (II) with CIR parameters $(a, b, \sigma_r, \rho_{s,r}) = (0.08, 0.8, 0, 0)$, we find that this deterministic model overstates the price most when volatility is low. For example, the bias is 36.4 cents (0.35%) when volatility is 20%. This is the case when hazard rates are constant but for equity linked hazard rates, the price difference increases with volatility. For instance, with the assumption of deterministic interest rates the prices rise to 57.7 cents (0.52%). More detailed results are available from the authors on request.

3.3. *Effect of default and recovery assumptions*

We now extend the empirical analysis to both reduced form and "blended-discount" default models, also changing the recovery assumptions and the assumptions regarding stock price behavior upon default. Again, volatility is still assumed to be constant and no notice period applies.

Assuming the face value is recovered and hazard rates are constant, Table 4 reports prices and hedge ratios for different levels of $p(0)$, R and η.[9] The notional instrument is 5-year CB with 8% semi-annual coupon, callable at a clean price of $170.0 in 3 years and putable at $109.0 in 3 years, and the common share pays no dividends. The bond is at the money (i.e., the conversion ratio is unity) and for the hazard rate model we set $\alpha = -1.0$. We set $\sigma_s = 20\%$ and the CIR parameters are $(a, b, \sigma_r, \rho_{s,r}) = (0.08, 0.8, 0.275, -0.3)$.

For the parameters chosen, the price sensitivity to $p(0)$ and to η is nearly identical, being virtually insensitive to the face recovery rate when $R \leq 0.6$. The

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[9]These may be benchmarked against the two-factor model prices of Tsiveriotis and Fernandes [46] which assume $\eta = 0$ and a credit spread of $p(1 - R)$.

Table 4. Effect of face recovery rate, stock loss rate, and hazard rate. Cases: variable η ($R = 0$, $p = 0.02$), variable R ($\eta = 0.3$ $p = 0.02$), variable p ($\eta = 0.3$, $R = 0.0$).

η	0.0	0.2	0.3	0.4	0.6	0.8	1.0
Price	134.889	134.408	134.173	133.944	133.499	133.071	132.657
Δ	0.712	0.720	0.724	0.728	0.736	0.743	0.750
Γ	6.07e−3	5.98e−3	5.93e−3	5.88e−3	5.78e−3	5.68e−3	5.57e−3
R	0.0	0.2	0.3	0.4	0.6	0.8	1.0
Price	134.173	134.173	134.174	134.175	134.282	134.905	136.063
Δ	0.724	0.724	0.724	0.724	0.714	0.690	0.670
Γ	5.93e−3	5.93e−3	5.94e−3	5.95e−3	6.45e−3	7.06e−3	6.92e−3
p	0.00	0.01	0.02	0.03	0.04	0.06	0.08
Price	136.805	135.439	134.173	133.001	131.909	129.895	128.082
Δ	0.671	0.699	0.724	0.746	0.767	0.804	0.836
Γ	7.13e−3	6.49e−3	5.93e−3	5.42e−3	4.97e−3	4.18e−3	3.51e−3

hedge ratio behavior is more illuminating: a high R reduces the delta significantly. At the extreme, if 100% of face value is recovered on default, 5.4% less need be invested in the stock hedge portfolio, compared to a zero recovery assumption. However a more realistic assumption, that only 30% is recovered, leaves delta unchanged. A low stock loss rate also reduces delta significantly; in particular, assuming the stock price jumps to zero gives a delta 3.8% higher than when the stock price is unaffected by default ($\eta = 0$). Compared to the default-free case ($p(0) = \eta = 0$), in the defaultable model we see a decrease in gamma. Figure 2 displays the price
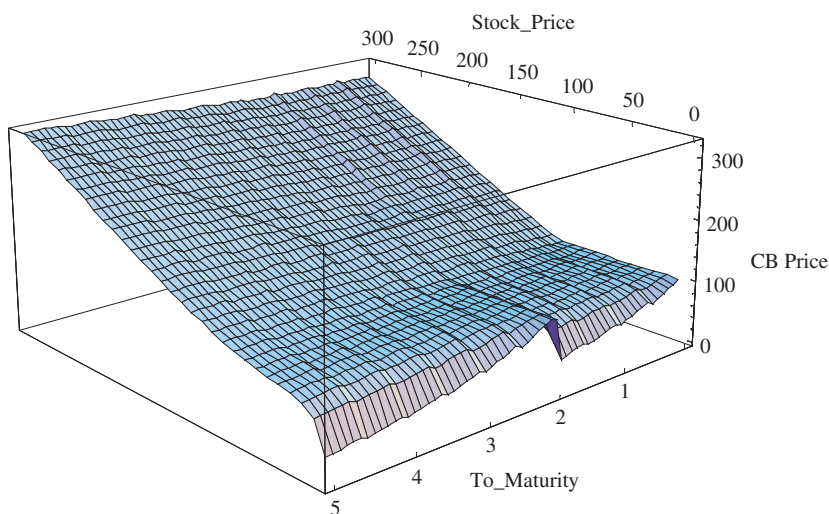

Fig. 2. Evolution of price of 5 year at the money CB with 8% semi-annual coupon, callable at $170 in 3 years and putable at $109.0 in 3 years.

surface evolution in time when $\eta = 0.8$, $R = 0$ and $p(0) = 0.02$. Notice that as $S \to 0$ the CB price $\to 0$ as its bond floor collapses due to the rapid rise in $p(S)$.

The choice between face and market value for recovery has only a small influence on prices and hedge ratios. Accurate assumptions about stock loss and recovery rates are far more important. To see this, Table 5 gives various prices and hedge ratios for a 7-year CB with 6% semi-annual coupon, callable at a clean price of $140.0 in 3 years and putable at $103.0 in 3 years, with a conversion ratio of 0.7. The equity and interest rate diffusion parameters are $\gamma = 0.01$, $\sigma_S = 50\%$ and $(a, b, \sigma_r, \rho_{s,r}) = (0.06, 0.8, 0.25, -0.3)$. The CB is priced using the reduced form model assuming recovery rates ranging from 0 to 0.5 of either the face value or the market value of the bond portion. In Table 5, these prices are denoted P_F and P_B respectively and similar subscripts are used for the hedge parameters. As expected, $P_F > P_B$ and, whilst the price difference $P_F - P_B$ increases with R, it is not as significant as some of the other prices effects shown here.

The prices and hedge ratios listed in the columns headed $\eta = 0$ in Table 5 are directly comparable with the Tsiveriotis and Fernandes (TF) model prices and hedge ratios with credit spread $s = p(1 - R)$. The TF results are shown in the columns headed s. With p fixed at 0.04, the credit spread decreases monotonically with as R increases. Note that the reduced form and TF models yield similar prices for high recovery levels but for low R (i.e., higher credit spreads) the prices and hedge ratios are *significantly* different. The reduced form prices are much higher than the TF prices for low values of R but marginally lower than TF prices when R is higher (≥ 0.5 here). The difference between the two model prices increases with the credit spread. In this example the price difference peaks at $4.21: the reduced from prices are 4% greater than the TF prices when the credit spread is 0.04. Marked differences in the hedge ratios are also observed. The reduced form deltas are significantly lower than the TF deltas, especially for high credit spreads; the TF model gammas are very small (of the order of 10^{-4}) and this is not the case for reduced form gammas.

Now consider the effect of changing the stock loss rate η. Table 5 shows that while CB prices are decreasing in η when the recovery rate is low, they increase with η when recovery rates are high. Evidently for sufficiently low recovery rates the drop in value of the conversion option in the default portfolio dominates but when the recovery rate is sufficiently high, it becomes optimal to not convert into defaulted stock, and then a higher stock loss rate increases the price.[10] The behavior of the prices P_F and P_B with respect to the stock loss rate is illustrated in Fig. 3. Observe that the turning point at which the price is increasing in the stock loss rate

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[10]This interesting feature can be explained by recalling that the time derivative V_t in the valuation equations is increasing in the spatial term $p\eta S V_S$ and decreasing in the defaulted portion of the portfolio $p\{\text{Max}(\kappa S(1 - \eta), RX)\}$. At low R, the default portfolio portion is dominated by the $\kappa(1 - \eta)$ term where initially the fall with η dominates the rise from the $p\eta S V_S$ term. With high R, the default portfolio portion is dominated by the RX term, so a high η (which decreases the $\kappa S(1 - \eta)$ term) has less effect while the positive impact comes through the $p\eta S V_S$ term.

Table 5. Comparison with TF model.

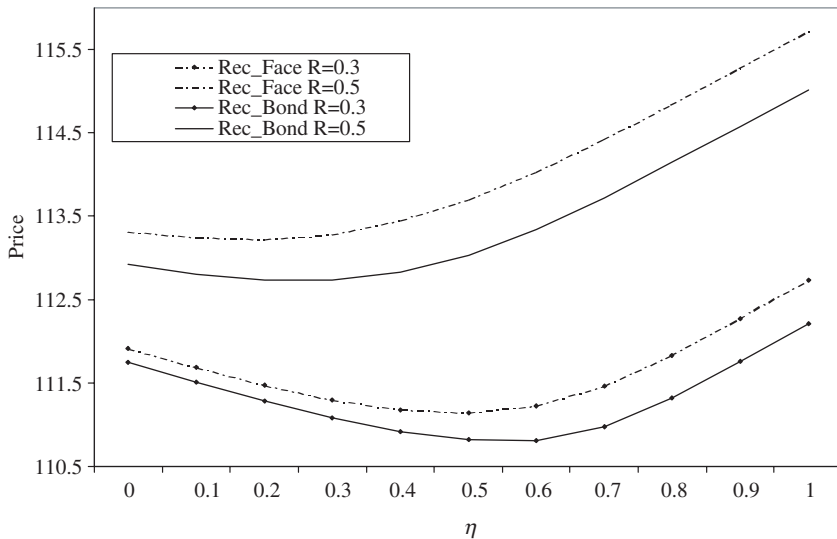|	R = 0.0				R = 0.1				R = 0.2				R = 0.3				R = 0.5			
	s	η			s	η			s	η			s	η			s	η		
	0.04	0	0.3	1	0.036	0	0.3	1	0.032	0	0.3	1	0.028	0	0.3	1	0.02	0	0.3	1
P_F	**107.2**	111.41	110.52	108.40	**108.52**	111.43	110.54	109.83	**109.76**	111.56	110.76	111.28	**111.06**	111.91	111.30	112.73	**113.39**	113.32	113.28	115.71
P_B		111.41	110.51	108.40		111.41	110.53	109.63		111.49	110.67	110.90		111.75	110.08	112.21		112.92	112.73	115.02
Δ_F	**0.564**	0.531	0.530	0.533	**0.550**	0.530	0.530	0.521	**0.537**	0.527	0.525	0.509	**0.523**	0.519	0.513	0.497	**0.492**	0.492	0.478	0.473
Δ_B		0.531	0.530	0.533		0.531	0.530	0.523		0.529	0.527	0.513		0.523	0.519	0.502		0.501	0.489	0.480
Γ_F	ε	−0.43	1.39	4.51	ε	−0.23	1.61	6.56	ε	0.96	3.38	8.59	ε	3.52	6.73	10.61	ε	11.41	15.79	14.56
Γ_B		−0.43	1.40	4.51		−0.37	1.47	6.04		0.22	2.44	7.67		1.84	4.72	9.39		8.22	12.60	13.11

Fig. 3. Relationship between price and stock loss rate for recovery of face value or bond value.

η occurs at a higher value of η when the recovery rate is lower. The difference in price when the recovery assumption changes from reduced form to blended discount is marginal at the start, but rises to 69 cents (0.6%) in the (possibly unrealistic) case of $R = 0.5$. Even for $R = 0.3$ and $\eta = 1$, the discrepancy is significant (52 cents, or 0.47%). This difference is even higher for longer maturity bonds. Estimates of both delta and gamma are affected, especially gamma which differs by up to 25%. Thus the trader is significantly over-hedged under the recovery of face value assumption if what is actually received is a proportion of market value upon default. These findings indicate that recovery assumptions can have a significant effect on prices and hedge ratios. Formulating the recovery aspect incorrectly could cost a trader 0.5% in price and double the estimate of gamma.

4. Uncertain Volatility and Delayed Calls

Many researchers have investigated issuers' call policies in the empirical literature, finding that they often wait until the conversion price is significantly higher than the effective call price plus the premium on the put option before issuing the call. Of the possible reasons for this "delayed call" phenomenon that have been proposed, in Sec. 2, we have focused on the uncertainty surrounding the common share price during the call notice period and the issuer's aversion to a sharp price decline [1, 5, 15, 25, 33]. For this reason, an uncertainty in trader's minds about the stock price volatility during the life of the CB was introduced as a means of modeling issuer's call policies.

Empirical results for the uncertain volatility model are now presented. First, Sec. 4.1 gives arbitrage-free bounds for prices and hedge ratios of representative CBs with various call and put features, assuming equity-linked hazard rates and stochastic interest rates. Then, Sec. 4.2 focuses on the call notice period aspect of the covenant, examining the call premiums that result with and without explicitly modeling the issuer's fear of a busted call. Finally, Sec. 4.3 examines a close to call CB, finding very significant call premiums arising from higher volatility uncertainty during the notice period.

4.1. *Uncertain equity volatility*

Two different contracts are considered to show how uncertainty in long-term equity volatility can affect the price and hedge ratios of CBs. Security A is out of the money, relatively long maturity (12 years), callable at 3 years but only at a price considerably exceeding the conversion price, and the volatility band is conservative. Security B is closer to call (in 1 year) and of 6-year maturity but we allow for a large uncertainty band for σ_S. Securities A and B both pay monthly coupons of 2% (annualized) and each have no notice period for the call. Table 6 summarizes the details of each security and Table 7 examines the effect of uncertainty in volatility on prices and hedge ratios by comparing certain volatility results with those from the uncertain volatility model. In the latter case, best and worst prices are reported in the last two columns of the Table 7. Stochastic interest rates are assumed, with CIR parameters $(a, b, \sigma_r, \rho_{s,r}) = (0.02, 0.8, 0.25, 0.2)$. We also report "delta-rho", the second derivative of the CB price with respect to S and r, denoted $\Delta\rho$ in the table.

Security A: Due to the negative gamma for this security, the certain volatility CB price is decreasing in volatility in the range. Most strikingly, even though the arbitrage-free price band provided by the best and worst uncertain volatility prices is quite narrow, not a single certain volatility price in the range comes to within $0.85 of these bounds. Thus the arbitrage-free range provides strong insurance for different volatility realizations in the stipulated range. Interestingly, in approximately 39% of the cases for best case pricing, the lowest volatility in the band was chosen in the numerical algorithm, alluding to the fact that the gamma frequently switches sign in the uncertain volatility model and that the delta-rho term becomes dominant in certain regions.[11]

Security B: The second instrument is closer to call (conversion price is $80 versus the call price which is $120) and of shorter term than security A. Although a wide volatility band $\sigma_S(t) \in [0.2, 0.4]$ is employed and we would expect broad

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[11]In a typical grid (with say 800 time-steps, 30 r-steps, and 200 S-steps) there are 4,884,000 volatility switching possibilities. Best volatility pricing with deterministic interest rates yields a price of $86.540, with 4,211 more switches to the "high" volatility case. Including stochastic interest rates appears to contribute 0.2% fewer switching decisions to the higher volatility regime as the $\Delta\rho$ term dominates the Γ term.

Table 6. CB contracts used in uncertain volatility examples.

Security A

Cash Flows				
Maturity	12.0	Face	$100	
κ	1.4	Coupon	0.02	
γ	0.02	Frequency	12	
σ_s	0.26–0.34			

Credit Model		Grid Dimensions	
S_{max}	$1000	p	0.04
r_{max}	0.20	η	0.2
N	2000	R	0.2
M	100		
L	20		

Call and Put Features		
Time	Price	Call/Put
3.0	160.0	Call
4.0	96.0	Put
8.0	101.0	Put

Security B

Cash Flows				
Maturity	6.0	Face	$100	
κ	0.8	Coupon	0.02	
γ	0.02	Frequency	12	
σ_s	0.20–0.40			

Credit Model		Grid Dimensions	
S_{max}	$2000	p	0.04
r_{max}	0.20	η	0.2
N	800	R	0.2
M	200		
L	4		

Call and Put Features		
Time	Price	Call/Put
1.0	120.0	Call
4.0	95.0	Put
varies	varies	Put

Security C

Cash Flows				
Maturity	7.0	Face	$100	
κ	1.0	Coupon	0.02	
γ	0.02	Frequency	2	
σ_s	0.22–0.28			

Credit Model		Grid Dimensions	
S_{max}	$1000	p	0.02
r_{max}	0.20	η	0.8
N	2000	R	0.2
M	100		
L	20		

Call and Put Features		
Time	Price	Call/Put
3.0 onwards	150.0	Call
5.0	102.0	Put
$7.0 - \tau$	∞	Call
($\tau = 1$–4 mths)		

Security D

Cash Flows				
Maturity	2.0	Face	$100	
κ	1.0	Coupon	0.04	
γ	0.02	Frequency	2	
σ_s	0.35–0.45			

Credit Model		Grid Dimensions	
S_{max}	$1000	p	0.02
r_{max}	0.20	η	0.8
N	2000	R	0.2
M	100		
L	20		

Call and Put Features		
Time	Price	Call/Put
1.0 onwards	110.0	Call
$2.0 - \tau$	∞	($\tau = 1$–4 mths)

Table 7. Comparison of certain and uncertain volatility models.

Security A

	Constant Volatility									Uncertain Volatility	
	0.26	0.27	0.28	0.29	0.30	0.31	0.32	0.33	0.34	Best	Worst
Price	85.891	85.829	85.753	85.665	85.566	85.458	85.343	85.222	85.097	86.8395	84.237
Δ	1.1891	1.197	1.204	1.211	1.217	1.223	1.228	1.233	1.237	1.2136	1.219
$\Gamma(\times 10^{-3})$	-4.33	-4.26	-4.16	-4.04	-3.91	-3.76	-3.61	-3.45	-3.29	-4.15	-3.46
ρ	-30.64	-30.10	-29.57	-29.06	-28.56	-28.08	-27.61	-27.15	-26.71	-30.42	-26.68
$\rho\Delta$	-0.153	-0.151	-0.148	-0.144	-0.139	-0.133	-0.127	-0.120	-0.113	-0.1486	-0.1305

Security B

$R = 0.2$
$\eta = 0.2$

	Constant Volatility									Uncertain Volatility	
	0.20	0.22	0.24	0.26	0.28	0.30	0.32	0.34	0.36	Best	Worst
Price	89.517	89.906	90.256	90.567	90.841	91.083	91.295	91.481	91.643	92.4135	89.1415
Δ	0.3631	0.373	0.382	0.389	0.396	0.404	0.4107	0.418	0.426	0.4279	0.3835
$\Gamma(\times 10^{-3})$	18.55	15.52	13.39	12.1	11.48	11.35	11.53	11.89	12.31	17.76	6.63
ρ	-66.87	-66.12	-65.42	-64.73	-64.03	-63.31	-62.56	-61.79	-61.00	-61.683	-64.67
$\rho\Delta$	0.348	0.331	0.320	0.315	0.313	0.313	0.313	0.312	0.311	0.3568	0.2233

Security B

$R = 0.5$
$\eta = 0.7$

	Constant Volatility									Uncertain Volatility	
	0.26	0.27	0.28	0.29	0.30	0.31	0.32	0.33	0.34	Best	Worst
Price	91.335	91.944	92.510	93.034	93.521	93.974	94.398	94.800	95.171	95.9875	91.2606
Δ	0.291	0.299	0.306	0.313	0.319	0.325	0.333	0.338	0.345	0.354779	0.295212
$\Gamma(\times 10^{-3})$	40.46	36.46	33.59	31.68	30.51	29.88	29.6	29.54	29.58	0.003127	0.003803
ρ	-72.05	-71.71	-71.38	-71.03	-70.65	-70.23	-69.79	-69.31	-68.81	-68.342	-71.6706
$\rho\Delta$	0.529	0.504	0.486	0.475	0.468	0.463	0.458	0.454	0.450	0.45029	0.5093

price bounds, the proximity to call is effective in narrowing the bounds. Thus the offsetting effect of the call feature is useful for generating viable trading prices from the model.[12] The difference between the best and worst price is $3.27, which is only just higher than for security A, even though there we assumed a tighter band for volatility ($\sigma_S(t) \in [0.26, 0.34]$). Again, the eventual volatility realization is well insured against. Even assuming a constant 40% volatility the best price (i.e., the trader's offered price) exceeds the constant volatility price by more than 50 cents, that is, 0.54%. We also look at the effect of recovery assumptions by increasing the face recovery rate to $R = 0.5$ of the face value and increasing the stock loss rate to $\eta = 0.7$ in the lower part of Table 7. Thus the bondholder gets the larger of $50 or the conversion value into the defaulted stock price after a 30% downward jump. While a larger amount of face value is recovered the stock price falls sharply on default. For high η we expect that the large volatility uncertainty band affecting the equity-linked hazard rate term $p(S)$ could possibly cause widening of arbitrage-free price bounds.[13] Arbitrage-free upper and lower prices in the band now differ by around $4.7 (nearly 5%). While larger than before, it should be borne in mind that the imposed volatility bands were quite wide.

In summary, the introduction of volatility uncertainty provides realistic arbitrage-free bounds for prices and hedge ratios of convertible bonds. The tightness of these bounds depends on the degree of uncertainty held over long-term volatility and how close the security is to call. Even with much uncertainty about volatility, call features provide a useful mechanism for generating viable trading prices that incorporate the trader's uncertain views on the stock price volatility.

4.2. *Call notice periods*

The above results were for CBs without call notice periods. Since call notice periods are most often included in CB covenants, we give examples that show how uncertain volatility during a call notice period can significantly increase the call premium. Security C is typical of many new issues. It is a 7-year semi-annual 4% coupon bond, with a high price ($150) no-notice call at 3 years, and a put at 5 years. In addition, it is unconditionally callable at any time after 3 years, with a clean price of $150. If it calls, the issuer is required to give a notice period of length τ. We consider three possibilities, effectively modeling three different securities, according as τ is one, two or four months. Since it not possible to call within τ months of expiry, in the basic terms for this security (see Table 6 above), we have followed convention and set the call price to infinity at time $7.0 - \tau$. The chosen CIR parameters are

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[12]Other means of narrowing the price bounds are possible. One of these is to combine the CB in an options hedge portfolio and price the portfolio. Due to volatility diversification effects the price bounds will be narrower. This idea was formalized by the Lagrangean uncertain volatility model due to Avellaneda and Paras [7].

[13]The volatility of the equity Brownian motion driven is equal to $\alpha\sigma S$. We refer to p. 5 in Andersen and Buffum [3] who have pointed this out.

$(a, b, \sigma_r, \rho_{s,r}) = (0.06, 0.8, 0.25, -0.3)$, so this time we assume a negative correlation between interest rates and stock price.

Table 8 reports the effect of uncertain volatility and length of notice period on both prices and hedge ratios. In this and all following tables the reported prices are the trader's *bid* price (i.e., the "worst price") with the notice period substitution asset being always "best priced".[14] Four cases are considered:

(a) certain volatility at 25%
(b) certain volatility at 25% and uncertain volatility (18% – 32%) only during the notice period
(c) uncertain volatility (22% – 28%) throughout the life of the CB
(d) uncertain volatility (22% – 28%) with volatility becoming more uncertain (18% – 32%) during the notice period

a) *Certain volatility:* The first column of Table 8 assumes that the firm calls as soon as the conversion price reaches the dirty call price. Adding a one-month notice increases the price by between 40.4 cents (one-month notice) and 85.4 cents (for a four-month notice period). As important, the notice period increases both delta and rho, because the notice period effectively increases the expected lifetime by raising the optimal stock price at which to call. For a four-month notice period, Table 8 shows that, in addition to underestimating its price, the holder of the CB would be under-hedged in stock by nearly 1% if the notice period were not taken into account.

The optimal stock price S^* at which the issuer should call is shown in Fig. 4. With a four-month notice period, the *average* call premium for this example is 27.7% above the clean call price ($150). Note that this is in broad agreement with the call premia assumed in Asquith [4] and Bingham [11], who use 20% for a 30-day notice period. We also observe that the implied call premium can jump significantly (by up to 60%). Figure 4 reveals two notable things: first, S^* falls sharply just before a coupon date. The firm is willing to call at a much lower price (still exceeding the dirty call price, which is between 150 and 152 at any given time) as it can force the bondholder to give up accrued interest (or coupon) should he convert. This is colloquially referred to as the "screw clause". Second, the issuer will never call after the last coupon has been paid. This is due to the fact that the cash-flow advantage is in the issuer's favor — it is no longer has to pay coupons but it would have to pay dividends if the investor converts due to the call. There is also no rationale for offering the substantially higher dirty call price as a floor when the bondholder is entitled only to the face value as floor without the call.

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[14] We report only the buyer's price as the seller's case is identical and has been discussed earlier. The substitution asset is "best priced" to maximize the value of the put option that the CB issuer gives the bondholder for free if he calls the bond for early redemption. If volatility turns out to be high during the notice period the substitution asset will be very precious. As the issuer is averse to busted calls, it should price this asset in the most expensive way possible.

Table 8. The effect of uncertain volatility with call notice periods (far from call case).

τ	0	1 month				2 months				4 months			
Model	(a)	(a)	(b)	(c)	(d)	(a)	(b)	(c)	(d)	(a)	(b)	(c)	(d)
price	115.55	115.954	116.015	114.458	114.52	116.152	116.224	114.64	114.706	116.403	116.48	114.857	114.918
Δ	0.7497	0.7527	0.7533	0.7539	0.7546	0.7544	0.7551	0.7558	0.7567	0.7569	0.7579	0.7586	0.7598
Γ	0.00347	0.00331	0.00329	0.00369	0.00367	0.00324	0.00322	0.00362	0.00360	0.00315	0.00313	0.00354	0.00353
ρ	−40.825	−41.173	−41.213	−39.216	−39.248	−41.318	−41.353	−39.305	−39.178	−41.448	−41.457	−39.339	−39.315
$\rho\Delta$	0.4246	0.4063	0.4035	0.4420	0.4386	0.3973	0.3941	0.4323	0.4288	0.3859	0.3824	0.4210	0.4814
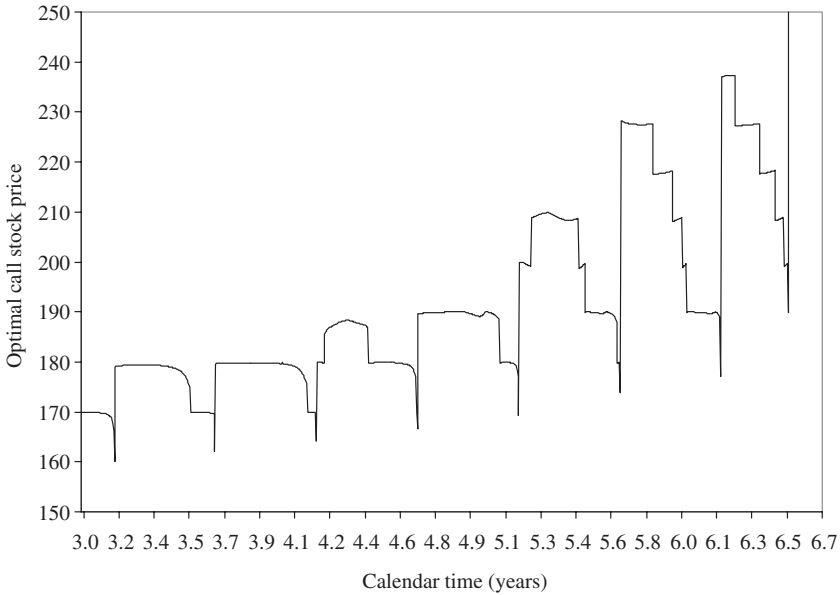
Fig. 4. Optimal call price with certain volatility (4 months notice period).

(b) *Uncertain volatility only during the notice period.* The convertible is priced assuming constant volatility of 25%. However the substitution asset is priced assuming an uncertain volatility band of [0.18, 0.32]. The prices and hedge ratios are reported in the columns headed (b) in Table 8. Introducing volatility uncertainty only during the notice period increases the bid price, but only marginally: the substitution asset affects prices only by 6 to 8 cents. That is because in this example, the clean call price ($150) is far from the current spot conversion price ($100). We shall see below that the uncertain volatility effects on closer to call CBs can be much more significant.

(c) *Uncertain volatility (same for substitution asset) and* (d) *uncertain volatility (higher for substitution asset).* Certain interesting properties are seen in the columns labeled (c) and (d) of Table 8. Although slightly higher bid prices are obtained when volatility uncertainty is greater for the substitution asset, uncertain volatility throughout the life of the CB tends to reduce the bid price (it is 1.2%–1.3% less than the constant volatility price) and this effect increases with the length of notice period. Volatility uncertainty also increases the hedge ratios, particularly when notice periods are short.

The main effect of notice period uncertain volatility for far from call CBs is on the call premium. Figure 5 illustrates the optimal call price for security C when a 2-month notice period applies. The lower curve corresponds to the case where the uncertain volatility band for pricing the substitution asset remains the same during the notice period. The average call premium compared to the clean call price of
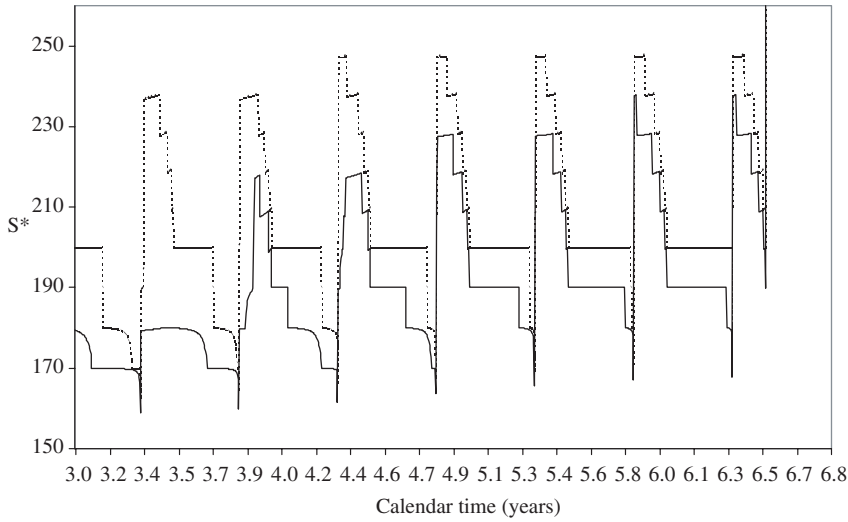
Fig. 5. Optimal call prices with uncertain volatility (far from call case).

$150 is 27.7%. The upper curve corresponds to the case where a wider uncertain volatility band is employed to price the substitution asset, as could be the case when the issuer fears a "busted call". The average call premium is much higher, at 38.4%. This explains the common practice of issuers that delay call until the conversion price is far greater than one would expect. Clearly the issuer's fear of a busted call, which is captured here by increasing volatility uncertainty during the notice period, can be an important factor.

4.3. *Close to call CBs*

Security C was far from call, so the substitution asset had little effect on prices, even though the call premium rose substantially when additional uncertainty in volatility was introduced during the notice period. When the CB is close to call, the instrument will be considerably more sensitive to assumptions regarding the notice period and the pricing of the substitution asset.

Security D is a 2-year bond, not putable but callable from the first year (see Table 6). Again we examine notice periods of 1, 2 or 4 months. The clean call price is $110.0 and our worst CB price is computed assuming that the stock price volatility lies in the range 35%–45%. Table 9 gives the bid prices and hedge ratios for security D under two different assumptions, according as we do or not assume a wider uncertainty band (35%–55%) for the pricing of the substitution asset. The price difference in this Table is in the range $0.4 to $0.52, which is comparable in size to the entire notice period effect without uncertain volatility. In fact, assuming a wider band of uncertainty during the notice period effectively *doubles* the average model call premium. It appears reasonable in this light that even moderately wider

Table 9. The effect of uncertain volatility with call notice periods (close to call case). I: Two regimes of Uncertain Volatility: (0.35, 0.45) for CB and (0.35, 0.55) for substitution asset; II: Uncertain Volatility (0.35, 0.45) throughout life of CB.

	1 month		2 months		4 months	
	I	II	I	II	I	II
Price	115.124	114.724	115.852	115.395	116.689	116.187
Δ	0.6604	0.6575	0.6699	0.6631	0.6818	0.6785
Γ	0.0065	0.0068	0.0061	0.0064	0.0060	0.0060
ρ	-39.143	-38.682	-40.351	-39.748	-41.153	-40.932
$\rho\Delta$	0.6713	0.6891	0.6336	0.6569	0.6003	0.6109
Average % diff of S* to clean call price	12.25%	5.52%	48.5%	22.5%	52.1%	23.5%
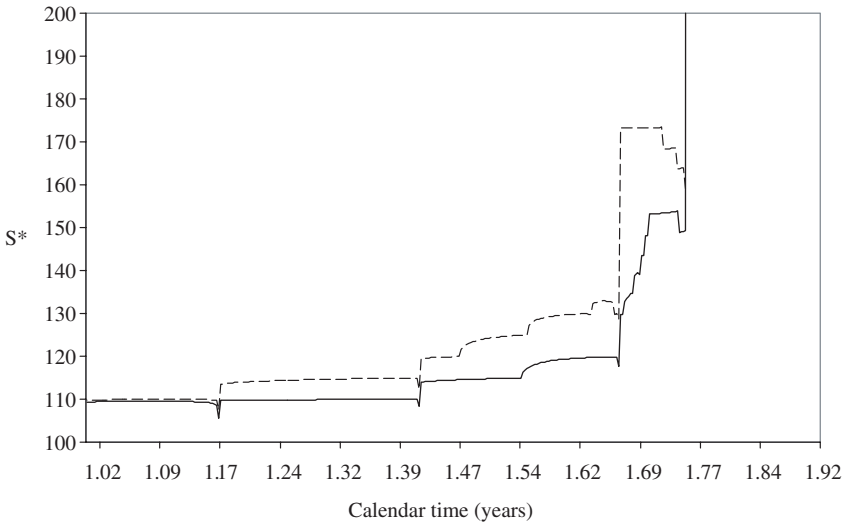
Fig. 6. Optimal call prices with uncertain volatility (close to call case).

uncertainty bands during the notice period have a substantial ability to capture observed call premia.

Figure 6 illustrates the optimal call prices for security D under the two volatility uncertainty assumptions and with a 2-month notice period. As in Fig. 5, the upper curve corresponds to the case where a wider band for volatility uncertainty is employed to price the substitution asset. Clearly the call premium increases as the CB approaches expiry. There is little incentive to call the bond for redemption when it has little time left to mature, because there is less time value to "kill" by forcing conversion, so the issuer will only consider calling the bond when the share price is significantly above the call price. The average call premium for the upper curve (compared to the clean call price) is 12.25% and for the lower curve is 5.5%.

Note that in this close to call example, the accrued interest effects are noticeable. The implied call policy actually falls *below* the effective call price by as much as $5.02. The issuer may call the bond for redemption when the stock price is below the effective call price because thereby it forces the bondholder to forfeit the accrued interest on the coupon.

5. Summary and Conclusions

This paper first examined the effect of call and put features, different assumptions about default behavior and recovery assumptions on the prices and hedge ratios of convertible bonds. We have employed a multi-factor model with stochastic interest rates and equity-linked hazard rates. The theoretically appealing framework of Ayache *et al.* [8] was easily extended to stochastic interest rates and, to value cross-currency convertibles, i.e., to include foreign exchange risk. Because the prices and hedge ratios of convertible bonds are quite sensitive to the call feature, much care has been taken when formulating this aspect of the contract. The PDE approach we employ is simple to implement and requires relatively few instruments for calibration. The model is implemented using unconditionally stable techniques, which are not subject to the numerical limitations associated with lattice methods. Then the empirical features of the model were illustrated for a range of realistic examples and some interesting properties have been identified.

Our empirical examples first examined the valuation effects of call and put features — and call notice periods in particular — when the stock price volatility is a known constant. Interest rate uncertainty was found to have a small but noticeable effect, especially when the conversion ratio is low and the correlation between stock price and interest rates is negative. Assumptions about recovery and the default process are crucial and these dominate the price effects of simple call and put features, although call and put features do have a more pronounced effect on prices and hedge ratios when there is a high probability that the issuer defaults. Prices are far lower under the assumption of equity-linked rather than constant hazard rates, and when recovery rates decrease. The main effect of reducing recovery rates is a significant decrease in stock deltas, whatever one assumes about stock loss and default behavior. Clearly an accurate modeling of default and recovery appears essential for traders to price these bonds properly and to hedge their positions effectively.

Because of their long maturity, convertible bonds prices and hedge ratios are very sensitive to assumptions on the stock price volatility, and this has been a main focus of this paper. Since the uncertainty about long-term volatility are particularly important, this paper has used the pioneering framework first introduced Avellaneda *et al.* [6] to include volatility uncertainty in the valuation model for convertible bonds. The addition of volatility uncertainty to the model allows traders to provide bounds for price and hedge ratios that are arbitrage-free, and these bounds will narrow as the security moves closer to call and/or as they become more certain

about volatility. Call and put features have a more pronounced effect on prices and hedge ratios when the stock price volatility is high and we argue that the issuer's delayed call policy can be intuitively explained by their uncertainty about stock volatility. The issuer's fear of a "busted call" during the call notice period can explain why the issuer delays calling until the stock price is very substantially above the effective call price, as was suggested by Ingersoll [33] and developed Asquith [4], Butler [15], Altintiğ and Butler [1], Grau *et al.* [25] and others. We explicitly model the issuer's fear of "busted calls" by increasing volatility uncertainty during the call notice period and show that the observed delay in issuer's call policies is, in fact, optimal in this framework. In summary, we have shown how uncertainty in the minds of the traders about the stock price volatility, and increased uncertainty during the call notice period, can be included in convertible bond valuation models. Within a multi-factor framework, with sophisticated modeling of interest rates, default and recovery, we have introduced volatility uncertainty as a mechanism to explain the delayed call features of issuer's optimal call policies.

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