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STOCHASTIC LOCAL VOLATILITY

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ABSTRACT
Local volatility models are commonly used for pricing and hedging exotic options consistently with a 'snap-shot' of Black-Scholes implied volatilities from traded vanilla options. However, there is substantial evidence that local volatility models fail to capture the proper dynamics of implied volatilities and that their hedging performance is poor, even for vanilla options. This is a consequence of the assumption of a deterministic spot volatility for the asset price process, which implicitly requires a static local volatility surface. In this paper, we re-visit the definition of local volatility and show that this assumption can be relaxed, while still preserving the attractive properties of local volatility models. Our approach differs from other papers on stochastic volatility since here we explicitly model the stochastic evolution of the local volatility surface rather than of the spot volatility. The resulting dynamics for implied volatility are shown to be equivalent to a 'market model' for implied volatility.

KEY WORDS
Option pricing and hedging, stochastic local volatility, implied volatility, smile-consistent models.

1. Introduction

A standard stylized fact in volatility theory is that the empirically observed 'smile' and 'skew' shapes in Black-Scholes implied volatilities contradict the model assumptions. Consequently many 'smile consistent' models have been developed (see Skiadopoulos [1] for a survey). Stochastic volatility models represent the spot volatility or variance as a diffusion or jump-diffusion process that is correlated with the underlying asset (e.g. Merton [2], Hull and White [3] and Heston [4]). Local volatility models derive spot and forward volatilities that are consistent with a 'snap-shot' of implied volatilities at a particular time (e.g. Dupire [5], Derman and Kani [6] and Rubinstein [7]). Stochastic and local volatility models have thus been regarded as two alternative and competing approaches to the same unobservable quantity, the spot volatility of the underlying asset.[1] However, several papers have tested the delta hedging performance of local volatility models and the general finding is that they perform even worse than the Black-Scholes model! For this reason the usual conclusion is that the assumption of a deterministic spot volatility is too restrictive and that stochastic volatility models are more realistic (e.g. Dumas *et al* [8]).

In fact, these two approaches are not inconsistent, but the few attempts to unify them into a single theory have not been much developed by further research. The heart of the problem is the assumption of a deterministic spot volatility that is imposed by most local volatility models. However, such assumption is not actually necessary for a local volatility model. This was recognized by Dupire [9] and Kani *et al* [10] who define the local variance (i.e. the square of the local volatility) as the expectation of the future spot variance conditional on a given asset price level. More specifically, at time $t_0 < t$ the local volatility function is

$$\sigma_{LV}^2(t,S) = E^0[\sigma^2(t,S(t),\mathbf{x}(t)) | S(t)=S] \tag{1}$$

where: E^0 denotes the expectation under the risk-neutral probability conditional on a filtration $\Im_0$ which includes all information up to time t_0; and $\mathbf{x}(t) = \{x_1, ..., x_n\}$ is a vector of all sources of uncertainty that influence the spot volatility process at time t, other than the asset price $S(t)$. Therefore the local volatility function is a deterministic function of time t and the future asset price even when the spot volatility is stochastic.[2] Note that $\mathbf{x}(t)$ is very general. It can be any 'arbitrage-free' set of continuous stochastic processes. Hence this definition of local volatility is consistent with any univariate diffusion stochastic volatility model in the literature (e.g. Hull and White [3] and Heston [4]). For this reason, Dupire [9] named model (1) the 'unified theory of volatility'.

[1] The 'spot' volatility is the diffusion coefficient in the underlying asset price process. Alternative terms are 'instantaneous' volatility or 'process' volatility.
[2] An analogy with the Heath-Jarrow-Morton model for interest rates is enlightening. The spot variance and local variance can be seen as analogous to the spot interest rate and the forward rate in the HJM model, so that the local volatility surface is the analogue to the forward yield curve.

Traditionally, local volatility models have ignored this extra uncertainty in spot volatility. But if spot volatility is deterministic, the model is incomplete. There is nothing in the model that tells one how to model local volatility dynamics. Indeed, the assumption of a deterministic spot volatility is inconsistent with *any* dynamics for local volatility. This could explain the poor empirical results on the hedging performance of local volatility models.

In this paper we extend the traditional view of local volatility by introducing explicit, stochastic dynamics for the parameters of the local volatility function. We call this more general model a 'stochastic local volatility' model to distinguish it from the incomplete model, which we term 'deterministic local volatility'. At each trading date a parameterized local volatility function is calibrated to a smile surface from market prices of European calls and puts, as usual. But we show that with stochastic local volatility the implied volatility dynamics requires an adjustment factor that depends on the degree of uncertainty in the local volatility parameters and on their correlation with the underlying price. In fact the stochastic local volatility model is equivalent to the 'market model' of implied volatility that has recently been studied by Schönbucher [11], Ledoit *et al* [12] and others.

The remainder of this paper is as follows: Section 2 introduces stochastic local volatility (SLV); Section 3 examines the dynamics of the SLV price of a contingent claim; Section 4 derives the dynamics of implied volatilities and proves the duality between the SLV model and the market model of implied volatilities; and Section 5 summarizes and concludes.

2. Stochastic Local Volatility

The popular definition of local volatility assumes the underlying asset price process follows a geometric Brownian motion with *deterministic* spot volatility $\sigma(t, S)$:

$$dS = (r - q)Sdt + \sigma(t, S)SdW_S \tag{2}$$

In this case, the local volatility is defined as equal to the spot volatility $\sigma(t, S)$ at any future time t and asset level S in the future. Then, the celebrated 'Dupire's equation' shows how the local volatility function $\sigma_{LV}(t, S)$ – and hence the spot volatility – may be uniquely determined from a surface of market prices $f(T, K)$ of standard European options with different strikes and maturities, as:

$$\sigma_{LV}^2(t,S)\Big|_{t=T,S=K} = 2\left(\frac{\partial f}{\partial T} + rK\frac{\partial f}{\partial K}\right)\Big/ K^2\frac{\partial^2 f}{\partial K^2} \tag{3}$$

where r denotes a constant risk-free interest rate and the dividend yield q has been assumed zero.[3]

However, since Dupire's equation requires a continuum of traded options prices, direct computation of the local volatility function (3) using finite difference methods is problematic. The local volatility surface can be very irregular and sensitive to the interpolation methods used between quoted option prices and their extrapolation to boundary values. Hence some of the more recent work on local volatility has focused on the use of parametric forms for local volatility functions. In this case the local volatility function is calibrated by changing parameters so that some distance metric between model prices and market prices is minimized and it may not fit quoted prices exactly (e.g. Dumas *et al* [8], Brigo and Mercurio [13] and Alexander [14]).

In these models, at any point in time t_0 the values $\mathbf{v}(t_0) = \{v_1(t_0), \ldots, v_n(t_0)\}$ for the local volatility parameters are calibrated to the current implied volatility surface. The underlying asset price process assumed at time t_0 is then:

$$dS = (r - q)Sdt + \sigma(t, S; \mathbf{v}(t_0))SdW_S \tag{4}$$

for all $t > t_0$ and $\mathbf{v}(t_0)$ is known at time t_0. Since the spot volatility is still deterministic, the local volatility is:[4]

$$\sigma_{LV}^2(t, S; \mathbf{v}(t_0)) = \sigma^2(t, S; \mathbf{v}(t_0)) \tag{5}$$

From henceforth this is referred to as the 'deterministic' local volatility model (DLV), since it assumes a deterministic spot volatility.

We have stressed that the local volatility will be sensitive to the calibration at time t_0. When at time $t_1 > t_0$ the model is re-calibrated, we will have:

$$\sigma_{LV}^2(t, S; \mathbf{v}(t_1)) = \sigma^2(t, S; \mathbf{v}(t_1)) \quad \text{for all } t > t_1$$

and this can of course differ from (5) as long as $\mathbf{v}(t_1) \neq \mathbf{v}(t_0)$. In fact, the dynamics of the local volatility will be stochastic if the calibrated parameters $\mathbf{v}(t)$ are stochastic.

Let us now assume that all the uncertainty in the random variables $\mathbf{x}(t)$ in (1) can be captured by the parameters $\mathbf{v}(t) = \{v_1(t), \ldots, v_n(t)\}$ of the local volatility model. Then the spot variance $\sigma^2(t, S(t); \mathbf{v}(t))$ at time t_0 is a function of t, S and $\mathbf{v}(t)$ and these parameters are stochastic variables because they are not calibrated to the market until some future time $t > t_0$. Note that only in the special case that the parameters $\mathbf{v}(t)$ are constant and equal to $\mathbf{v}(t_0)$ do we have the deterministic local volatility model (5) above. In general, when we allow $\mathbf{v}(t)$ to evolve stochastically, we have, at time t_0:

$$\begin{aligned}\sigma_{LV}^2(t,S) &= E^0\left(\sigma^2(t, S(t); \mathbf{v}(t)) \middle| S(t) = S\right) \\ &= \int_{\Omega_t} \sigma^2(t, S; \mathbf{v}) h_t(\mathbf{v} \mid S) d\mathbf{v}\end{aligned} \tag{6}$$

In (6), as in (1), the expectation is conditional on a filtration $\Im_0$, which includes all information up to time t_0. The integration is over Ω_t, the space of all 'arbitrage-free' values for $\mathbf{v}(t)$ and $\mathbf{v} \in \Omega_t$ is a realization of $\mathbf{v}(t)$.[5] Finally

[3] Note that in the special case that $\sigma(t, S)$ is not a function of $\mathbf{x}(t)$ the later definition of local volatility (1) reduces to this early definition of local volatility, and Dupire's equation is consistent with the general 'forward equation' derived by Kani *et al* [10]. That is, the definition (1) for local volatility nests the early definition. See also Skiadopoulos [1].

[4] Equation (5) is derived from (1) using $\mathbf{x}(t) \equiv \mathbf{v}(t) = \mathbf{v}(t_0)$.

[5] Note that, depending on the functional form assumed for the spot variance, some values for $\mathbf{v}(t)$ can introduce arbitrage opportunities. Therefore, those values are excluded from Ω_t in an arbitrage-free set-up.

$h_t(\mathbf{v}|S)$ denotes the multivariate density of $\mathbf{v}(t)$ conditional on a given S at time t, and given $\Im_0$.[6] With this definition the local volatility function $\sigma^2_{LV}(t,S)$ has an implicit dependence on the future parameters $\mathbf{v}(t)$ – and their past values, through the filtration $\Im_0$ – so the local volatility becomes stochastic. We call (6) the 'stochastic local volatility' (SLV) model.

Although the local volatility surface derived from a DLV model can fit the current smile, the assumption of a deterministic spot volatility is unrealistic. So in the same way that stochastic volatility models extend the Black-Scholes (BS) assumptions to a more realistic volatility process, deterministic local volatility models must be extended to account for the uncertainty in the future values of their parameters.

3. Option Price Dynamics

Now assume the asset price follows a geometric Brownian motion under the risk-neutral measure Q:

$$dS = (r-q)Sdt + \sigma(t,S;\mathbf{v}(t))SdW_S \qquad (7)$$

where the spot volatility $\sigma(t, S; \mathbf{v}(t))$ is a continuous process satisfying usual regularity conditions. Assume also that the continuously-compounded risk-free rate r and dividend yield q are constant. The spot volatility is stochastic since $\mathbf{v}(t) = \{v_1(t), v_2(t), \ldots v_n(t)\}$ is a vector of stochastic parameters that are correlated with the asset price $S(t)$ and with each other. Assume the risk-neutral dynamics for each parameter v_i in $\mathbf{v}(t)$ are as follows:

$$dv_i = \alpha_i(t,S,\mathbf{v})dt + \beta_i(t,S,\mathbf{v})dZ_i \qquad (8)$$

$$dZ_i = \rho_{i,S}(t,S,\mathbf{v})dW_S + \sqrt{1-\rho^2_{i,S}(t,S,\mathbf{v})}dW_i$$

with $dZ_i dZ_j \overset{a.s.}{\to} \rho_{i,j}(t,S,\mathbf{v})dt$ and $dW_i dW_S \overset{a.s.}{\to} 0$ for $i, j \in \{1, 2\ldots n\}$, satisfying almost surely and for all $T > t_0$:

$$\int_{t_0}^{T} |\alpha_i(t,S,\mathbf{v})| dt < \infty \quad \text{and} \quad \int_{t_0}^{T} \beta_i^2(t,S,\mathbf{v})dt < \infty$$

so that $\rho_{i,j} \in [-1, 1]$ is the correlation between variations in v_i and v_j and $\rho_{i,S} \in [-1, 1]$ is the correlation between variations in v_i and S. Together (7) and (8) provide the full specification of the SLV model.

Denote the local volatility price, calibrated at time $t_0 < t$, of a contingent claim by $f_L = f_L(t, S(t); \mathbf{v}(t) | \Im_0)$. So $\mathbf{v}(t_0)$ is included in the filtration $\Im_0$. Since $\mathbf{v}(t)$ contains the future parameters of a *deterministic* local volatility model, it follows that the claim price f_L must satisfy the following differential equation at each time $t > t_0$:[7]

$$\frac{\partial f_L(t,S;\mathbf{v})}{\partial t} + (r-q)S\frac{\partial f_L(t,S;\mathbf{v})}{\partial S} + \tfrac{1}{2}\sigma^2(t,S;\mathbf{v})S^2\frac{\partial^2 f_L(t,S;\mathbf{v})}{\partial S^2} = rf_L(t,S;\mathbf{v}) \qquad (9)$$

where the filtration $\Im_0$ has been omitted for convenience. Equation (9) only holds locally, i.e. assuming the DLV model is re-calibrated at each time t. But since the calibrated parameters are likely to be different at each re-calibration we can assume $\mathbf{v}(t)$ is stochastic and defined as in (8) above.

Alexander and Nogueira [15] show that under assumptions (7) and (8), the risk-neutral dynamics of the model price $f_L(t, S(t); \mathbf{v}(t)|\Im_0)$ are:

$$df_L = rf_L dt + \left(\sigma S\frac{\partial f_L}{\partial S} + \sum_i \beta_i\rho_{i,S}\frac{\partial f_L}{\partial v_i}\right)dW_S + \sum_i \beta_i\sqrt{1-\rho^2_{i,S}}\frac{\partial f_L}{\partial v_i}dW_i \qquad (10)$$

at every time $t \in (t_0, T)$ and the coefficients in (8) must satisfy the following drift condition for every $t \in (t_0, T)$:

$$\sum_i\left(\alpha_i\frac{\partial f_L}{\partial v_i} + \sigma S\beta_i\rho_{i,S}\frac{\partial^2 f_L}{\partial v_i\partial S} + \tfrac{1}{2}\sum_j \rho_{i,j}\beta_i\beta_j\frac{\partial^2 f_L}{\partial v_i\partial v_j}\right) = 0$$

Hence the two model price dynamics are *not* the same because only the SLV model takes account of the uncertainty in future values of the model parameters. Dynamics (10) contrast with the dynamics for the option price from DLV models. In effect, by assuming $\beta_i = 0$ for every parameter v_i in the SLV model above (i.e. no stochastic behavior for the model parameters), it is easy to see that dynamics (10) collapse to the standard 'deterministic' local volatility option price dynamics:

$$df_L = rf_L dt + \sigma S\frac{\partial f_L}{\partial S}dW_S \qquad (11)$$

Therefore, if the calibrated model parameters are stable over time (i.e. $\mathbf{v}(t) = \mathbf{v}(t_0)$ for all $t > t_0$, or equivalently, $\alpha_i = \beta_i = 0$ for all v_i) one can claim that the DLV risk-neutral price dynamics (11) are accurate. Otherwise, if either $\alpha_i \neq 0$ or $\beta_i \neq 0$ (or both), the dynamics (11) are incomplete.

However when the parameters $\mathbf{v}(t)$ of the local volatility model are stochastic with less than perfect correlation with asset price movements, the claim price has multi-factor dynamics (10) with one Brownian motion from the underlying asset price dynamics (7) and another Brownian motion for each stochastic parameter in the model, contrasting with the DLV model dynamics (11). But of course different price dynamics imply different implied volatility dynamics and it is likely that the implied volatility dynamics derived from DLV models will be incorrect, as shown in the next section. This result concords with Hagan *et al* [16], who claim that local volatility models fail to capture the proper dynamics of implied volatilities.

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[6] From probability theory, we know that $h_t(\mathbf{v}|S)$ is related to the joint density of S and $\mathbf{v}$ by $h_t(\mathbf{v}|S) = h_t(\mathbf{v}, S)/g_t(S)$, where $g_t(S)$ is the unconditional density of S at time t.

[7] Within a deterministic local volatility model, $\mathbf{v}(t)$ is assumed constant, hence f_L can be expressed as a function of t and S only. Then (9) follows from application of Ito's lemma and the standard risk-neutrality argument.

4. Implied Volatility Dynamics

Recent work of Dupire [17] derives a general relationship between local volatilities and BS implied volatilities, in which implied volatilities are the square root of gamma-weighted averages of local variances. This raises the question of duality between a stochastic local volatility model and a stochastic implied volatility model. This section formalizes the duality result by deriving an explicit relationship between the stochastic local volatility price dynamics for vanilla options and the evolution of the associated implied volatility. For a vanilla European option with strike K and maturity T, the local volatility price of this option at time t when the asset price is S is denoted by $f_L(K, T; t, S, \mathbf{v})$. Note that we term f_L the 'local volatility' price, with no distinction between the deterministic and the stochastic volatility model price. Assuming the DLV model is continuously re-calibrated, they will be identical at every point in time for a European option.

Similarly, when the BS implied volatility is θ, we denote the BS price of this option at time t when the asset price is S by $f_{BS}(K, T; t, S, \theta)$. We define the *market* implied volatility $\theta_M = \theta_M(K, T; t, S)$ as that θ such that the BS model price equals the observed market option price. Since market prices are observable, market implied volatilities are observable.

Now assume that the local volatility model is calibrated to an implied volatility surface at each time t. Then the *local* implied volatility $\theta = \theta(K, T; t, S, \mathbf{v})$ is defined by equating the local volatility price to the BS price:

$$f_L(K,T;t,S,\mathbf{v}) = f_{BS}(K,T;t,S,\theta(K,T;t,S,\mathbf{v})) \quad (12)$$

The following results are derived using local implied volatilities, not market implied volatilities. That is, we derive the relationship between a stochastic local volatility function and the associated local implied volatilities on the assumption that the parametric local volatility model can fit market options prices on any day with acceptable accuracy.

To prove the theorem we first need two lemmas. These focus on the sensitivities of the local implied volatility surface $\theta(K, T; t, S, \mathbf{v})$ to changes in t, S and $\mathbf{v}$. These depend on the dynamics of the implied volatility surface – they cannot be derived from a single 'snap-shot' of the surface. There is a large empirical literature on the implied volatility sensitivity to S (e.g. Derman and Kamal [18], Skiadopoulos *et al* [19], Alexander [20] and others) but we shall take a theoretical approach here. We assume that the parameters of a local volatility model evolve stochastically as specified in (8) and we derive the implied volatility dynamics that are consistent with this.

Denote the BS model price sensitivities by $\delta_{BS} = \partial f_{BS}/\partial S$; $\gamma_{BS} = \partial^2 f_{BS}/\partial S^2$; $\Theta_{BS} = \partial f_{BS}/\partial t$; $\Upsilon_{BS} = \partial f_{BS}/\partial\theta$; $\kappa_{BS} = \partial^2 f_{BS}/\partial\theta^2$ and $\Omega_{BS} = \partial^2 f_{BS}/\partial S\partial\theta$, and the DLV sensitivities by $\delta_L = \partial f_L/\partial S$, $\gamma_L = \partial^2 f_L/\partial S^2$ and $\Theta_L = \partial f_L/\partial t$.

Lemma 1

The local implied volatility function $\theta(K, T; t, S, \mathbf{v})$ has sensitivities to t, S and $\mathbf{v}$ given by:

$$\frac{\partial\theta(K,T;t,S,\mathbf{v})}{\partial t} = \frac{\Theta_L(K,T;t,S,\mathbf{v}) - \Theta_{BS}(K,T;t,S,\theta)}{\Upsilon_{BS}(K,T;t,S,\theta)}$$

$$\frac{\partial\theta(K,T;t,S,\mathbf{v})}{\partial S} = \frac{\delta_L(K,T;t,S,\mathbf{v}) - \delta_{BS}(K,T;t,S,\theta)}{\Upsilon_{BS}(K,T;t,S,\theta)}$$

$$\frac{\partial\theta(K,T;t,S,\mathbf{v})}{\partial v_i} = \frac{1}{\Upsilon_{BS}(K,T;t,S,\theta)} \frac{\partial f_L(K,T;t,S,\mathbf{v})}{\partial v_i}$$

$$\frac{\partial^2\theta}{\partial S^2} = \frac{1}{\Upsilon_{BS}}\left[\gamma_L - \gamma_{BS} - \kappa_{BS}\left(\frac{\partial\theta}{\partial S}\right)^2 - 2\Omega_{BS}\frac{\partial\theta}{\partial S}\right]$$

$$\frac{\partial^2\theta}{\partial S\partial v_i} = \frac{1}{\Upsilon_{BS}}\left[\frac{\partial^2 f_L}{\partial S\partial v_i} - \frac{\partial\theta}{\partial v_i}\left(\Omega_{BS} + \kappa_{BS}\frac{\partial\theta}{\partial S}\right)\right]$$

$$\frac{\partial^2\theta}{\partial v_i\partial v_j} = \frac{1}{\Upsilon_{BS}}\left[\frac{\partial^2 f_L}{\partial v_i\partial v_j} - \kappa_{BS}\frac{\partial\theta}{\partial v_i}\frac{\partial\theta}{\partial v_j}\right]$$

Proof: Differentiate (12) with respect to t, S and each v_i and apply the chain rule in the right-hand side whenever necessary. For instance:

$$\frac{\partial f_L}{\partial S} = \frac{\partial f_{BS}}{\partial S} + \frac{\partial f_{BS}}{\partial\theta}\frac{\partial\theta}{\partial S} \Rightarrow \frac{\partial\theta}{\partial S} = \frac{\delta_L - \delta_{BS}}{\Upsilon_{BS}}$$

$$\frac{\partial^2\theta}{\partial S^2} = \frac{\partial}{\partial S}\left(\frac{\delta_L - \delta_{BS}}{\Upsilon_{BS}}\right)$$

and so forth.

Lemma 2

Any local implied volatility $\theta(K, T; t, S, \mathbf{v})$ for an European option with strike K and maturity T must satisfy the following partial differential equation:

$$\left[\begin{array}{l} \dfrac{\partial\theta}{\partial t} + \left(r - q - \sigma^2\dfrac{d_2}{\theta\sqrt{\tau}}\right)S\dfrac{\partial\theta}{\partial S} + \\ \tfrac{1}{2}\sigma^2 S^2\left(\dfrac{\partial^2\theta}{\partial S^2} + \dfrac{d_1 d_2}{\theta}\left(\dfrac{\partial\theta}{\partial S}\right)^2\right) + \tfrac{1}{2}\dfrac{1}{\theta\tau}\left(\sigma^2 - \theta^2\right) \end{array}\right] = 0$$

with $\tau = T - t > 0$ and d_1 and d_2 as in the BS formula:

$$d_1 = \frac{\ln M}{\theta\sqrt{\tau}} + \tfrac{1}{2}\theta\sqrt{\tau} \qquad d_2 = d_1 - \theta\sqrt{\tau} \qquad M = \frac{Se^{-q\tau}}{Ke^{-r\tau}}$$

Proof: Subtract the BS PDE from equation (9), apply (12) and Lemma 1, and use the relationship between the BS sensitivities γ_{BS}, Υ_{BS}, κ_{BS} and Ω_{BS} defined above to simplify the expression.

Lemma 2 describes the dynamics of implied volatility that are consistent with any parametric local volatility model. Note that the differential equation in Lemma 2 has no partial derivative on the elements of $\mathbf{v}$ (but of course $\theta(K, T; t, S, \mathbf{v})$ is not independent of $\mathbf{v}$ because it depends on the spot volatility $\sigma(t, S; \mathbf{v})$). So even when, as assumed in DLV models, the local volatility parameters are constant, the implied volatility surface *does* move over time. But whilst these models are not inconsistent with movement in implied volatilities over time, problems arise because the permissible movements in implied volatility are very restricted (see e.g. Hagan *et al* [16]).

Now the main result of this section shows how the SLV model determines the dynamics of each implied volatility of strike K and maturity T. Indeed the dynamics of the entire implied volatility surface are governed by the same stochastic factors as those driving the local volatility and the option price:

Theorem 1
Under assumptions (7) and (8), the dynamics of $\theta = \theta(K, T; t, S, \mathbf{v})$, the local implied volatility for a European option with strike K and maturity T, are given by:

$$d\theta = \xi dt + \sigma S \frac{\partial\theta}{\partial S} dW_S + \sum_i \frac{\partial\theta}{\partial v_i}\beta_i dZ_i \qquad (13)$$

where the drift ξ is given by:

$$\xi = \tfrac{1}{2}\frac{1}{\theta(T-t)}\left(\theta^2 - \sigma^2\right) + \sigma\frac{d_2}{\theta\sqrt{T-t}}\psi - \tfrac{1}{2}\frac{d_1 d_2}{\theta}\eta^2$$

satisfying $\int_{t_0}^{T}|\xi|dt < \infty$ and ψ is related to the covariance between implied volatility and asset price movements:

$$\psi dt = d\theta dW_S = \left(\sigma(t,S;\mathbf{v})S\frac{\partial\theta}{\partial S} + \sum_i \beta_i\rho_{i,S}\frac{\partial\theta}{\partial v_i}\right)dt$$

and η^2 is the variance of the implied volatility process:

$$\eta^2 dt = d\theta d\theta = \left(\psi^2 + \sum_i\sum_j \beta_i\beta_j\left(\rho_{i,j} - \rho_{i,S}\rho_{j,S}\right)\frac{\partial\theta}{\partial v_i}\frac{\partial\theta}{\partial v_j}\right)dt$$

and all partial derivatives of θ are as in Lemma 1.

Proof: Dynamics (13) follow from a straightforward application of Ito's lemma to the dynamics of $\theta(K, T; t, S, \mathbf{v})$ w.r.t. t, S and $\mathbf{v}$, and of lemmas 1 and 2.

The following corollary, which simply re-writes (13) using only uncorrelated Brownian motions, is interesting because it shows that the SLV model is equivalent to the dynamic 'market model' for implied volatilities introduced by Schönbucher [11]:

Corollary 1
Assuming the vector $\boldsymbol{dW} = [dW_1, dW_2 \dots dW_n]$ has positive definite correlation matrix $\mathbf{\Sigma}$, the dynamics of the local implied volatility from Theorem 1 can also be expressed in terms of uncorrelated Brownian motions as:

$$d\theta = \xi dt + \psi dW_S + \sum_j \omega_j dW_j^* \qquad (14)$$

with ξ and ψ defined as in Theorem 1 and $dW_S dW_j^* = dW_i^* dW_j^* = 0$ for all $i \neq j$ almost surely, and:

$$\omega_j = \sum_{i=j}^{n}\beta_i\sqrt{1-\rho_{i,S}^2}\,A_{i,j}\frac{\partial\theta}{\partial v_i} \qquad (15)$$

where $A_{i,j}$ are the elements of the Cholesky decomposition $\mathbf{A}$ of the correlation matrix $\mathbf{\Sigma}$ with:

$$\Sigma_{i,j} = \frac{\rho_{i,j} - \rho_{i,S}\rho_{j,S}}{\sqrt{(1-\rho_{i,S}^2)(1-\rho_{j,S}^2)}} \quad \text{and} \quad \eta^2 = \psi^2 + \sum_j \omega_j^2 .$$

Proof: Follows from a standard application of Cholesky decomposition.

Apart from minor differences in notation, equation (14) is the same as equation (2.7) from Schönbucher [11] for the dynamics of a stochastic implied volatility with the drift term given by equation (3.7).[8] However, note that Schönbucher [11] models stochastic implied volatility for a *given* strike K and maturity T, whilst we begin with a stochastic local volatility model for which the dynamics (14) hold for *all* strikes and maturities simultaneously. In the 'market models' approach, an implied volatility (or implied variance) diffusion is defined for each strike K and maturity T. So if there are options for k strikes and m maturities in the market, the market model specifies mk diffusions, one for each traded option. But in the SLV approach, the smile surface is parameterized, with the number of parameters $n \ll mk$. Hence the SLV model reduces the probability space from $mk + 1$ random variables to only $n + 1$, including the asset price S.

Corollary 2
The correlation between the local implied volatility and the asset price movements is given by:

$$\rho_{\theta,S}(K,T;t,S,\mathbf{v}) = \frac{\psi}{\sqrt{\psi^2 + \sum_j \omega_j^2}} = \frac{\psi}{|\eta|} \qquad (16)$$

Proof: The correlation follows from (7) and (14):

$$\rho_{\theta,S} = \frac{Cov(d\theta, dS)}{\sqrt{Var(d\theta)Var(dS)}} = \frac{\psi\sigma S dt}{\sqrt{(\psi^2 + \sum_j \omega_j^2)dt(\sigma^2 S^2 dt)}} .$$

We have used the absolute value $|\eta|$ to stress that the denominator of (16) is strictly positive. The ω_j are defined in (15) and are non-zero unless the parameters are non-stochastic. Note that the local implied volatility and price will have perfect correlation (of ± 1, depending on the sign of the covariance ψ) if and only if $\omega_j = 0$ for all j, i.e. in the DLV model. Hence the local volatility is deterministic if and only if variations in implied volatility and the asset price are perfectly correlated.

5. Conclusion

Two approaches to modeling the Black-Scholes implied volatility smile or skew surface have been developed separately even though potential links between them were identified many years ago by Dupire [9] and Kani *et al* [10]. In the intervening years most research on stochastic volatility has specified a univariate diffusion or

[8] We believe there was a typing error in equation (3.3) from Schönbucher [11] for the variance of implied volatility, where the term γ^2 appears to be missing. Many thanks to Hyungsok Ahn of Commerzbank, London for drawing our attention to this.

jump-diffusion for the spot variance or volatility of the underlying asset. Likewise, most research on local volatility models has assumed a deterministic spot volatility function for the underlying asset price diffusion at a particular point in time, with no reference to the dynamic evolution of volatility.

Both approaches are incomplete, the former capturing the dynamic properties of volatility but only in a one-dimensional space, the latter focusing on the multi-dimensional aspects of volatility but ignoring its time-evolution. However recent developments of multivariate diffusions for implied volatility have extended the stochastic volatility approach to be consistent with the cross-section of implied volatilities as well as their dynamics. To concord with this view, the deterministic local volatility model, which implies only a deterministic evolution for implied volatility, requires generalization.

Following Dupire [9] and Kani *et al* [10] we regard the deterministic local volatility model as merely a special case of a more general stochastic local volatility model. That is, we define local volatility as the square root of the conditional expectation of a future spot variance that depends on $n + 1$ stochastic risk factors, viz. the underlying price plus n parameters of the local volatility function.

Hence we provide an explicit model of the stochastic evolution of a locally deterministic volatility surface over time. We have proved that this general stochastic local volatility model is equivalent to the market model for implied volatilities that was introduced by Schönbucher [11]. This important 'duality' result has shown that the stochastic and local volatility approaches can be unified within a single general framework. It is only when these approaches take a restricted view on volatility that they appear to be different.

There are a variety of possible applications for this model, for instance, to the analysis of implied volatility dynamics. But perhaps the most important application is to use the claim price dynamics (10) for pricing exotic options and/or to derive the correct local volatility hedge ratios for standard European options, as in Alexander and Nogueira [15].

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References:

[1] G. Skiadopoulos, Volatility smile consistent option models: A survey, *International Journal of Theoretical and Applied Finance*, *4*(3), 2001, 403-438.

[2] R. Merton, Option pricing when underlying stock returns are discontinuous, *Journal of Financial Economics*, *3*, 1976, 125-144.

[3] J. Hull & A. White, The pricing of options on assets with stochastic volatilities, *Journal of Finance, 42*, 1987, 281-300.

[4] S. Heston, A closed form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies, 6*(2), 1993, 327-343.

[5] B. Dupire, Pricing with a Smile, *Risk 7*(1), 1994, 18-20.

[6] E. Derman & I. Kani, Riding on a smile, *Risk, 7*(2), 1994, 32-39.

[7] M. Rubinstein, Implied binomial trees, *Journal of Finance, 69*, 1994, 771-818.

[8] B. Dumas, F. Fleming & R. Whaley, Implied volatility functions: empirical tests. *Journal of Finance, 53*(6), 1998, 2059-2106.

[9] B. Dupire, A Unified Theory of Volatility, Incomplete Working Paper, January 1996.

[10] I. Kani, E. Derman & M. Kamal Trading and hedging local volatility, *The Journal of Financial Engineering, 6*, 1997, 1233-1268.

[11] P.J. Schönbucher, A market model of stochastic implied volatility, *Philosophical Transactions of the Royal Society, Series A, 357*, 1999, 2071-2092.

[12] O. Ledoit & P. Santa-Clara, Relative pricing of options with stochastic volatility, *University of California-Los Angeles Finance Working Paper 9-98*, 1998 [now updated to 2002 with third co-author S. Yan].

[13] D. Brigo & F. Mercurio, Lognormal-mixture dynamics and calibration to market volatility smiles, *International Journal of Theoretical & Applied Finance 5*(4), 2002, 427-446.

[14] C. Alexander, Normal mixture diffusion with uncertain volatility: modelling short and long term smile effects, *Journal of Banking and Finance*, forthcoming.

[15] C. Alexander & L.M. Nogueira, Hedging with stochastic local volatility, *ISMA Centre Discussion Paper in Finance 2004-11*, University of Reading, 2004.

[16] P.S. Hagan, D. Kumar, A.S. Lesniewski, & D.E. Woodward, Managing smile risk, *Wilmott Magazine, 1*, 2002, 84-108.

[17] B. Dupire, Stochastic volatility modelling, *ICBI Global Derivatives Conference*, Nice, 2003.

[18] E. Derman & M. Kamal, The patterns of change in implied index volatilities, *Quantitative Strategies Research Notes*, Goldman Sachs, 1997.

[19] G. Skiadopoulos, S. Hodges & L. Clewlow, The dynamics of the S&P 500 implied volatility surface, *Review of Derivatives Research, 3*(3), 1999, 263-282.

[20] C. Alexander, Principles of the skew, *Risk 14*(1), 2001, S29-S32, also in Alexander Lipton (ed.), Exotic Options, *Risk Publications*, 2003, 57-64.