

# Closed Form Approximations for Spread Options

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**ABSTRACT** *This article expresses the price of a spread option as the sum of the prices of two compound options. One compound option is to exchange vanilla call options on the two underlying assets and the other is to exchange the corresponding put options. This way we derive a new closed form approximation for the price of a European spread option and a corresponding approximation for each of its price, volatility and correlation hedge ratios. Our approach has many advantages over existing analytical approximations, which have limited validity and an indeterminacy that renders them of little practical use. The compound exchange option approximation for European spread options is then extended to American spread options on assets that pay dividends or incur costs. Simulations quantify the accuracy of our approach; we also present an empirical application to the American crack spread options that are traded on NYMEX. For illustration, we compare our results with those obtained using the approximation attributed to Kirk (1996, Correlation in energy markets. In: V. Kaminski (Ed.), Managing Energy Price Risk, pp. 71–78 (London: Risk Publications)), which is commonly used by traders.*

**KEY WORDS:** spread options, exchange options, American options, analytical approximation, Kirk's approximation, correlation skew

## 1. Introduction

A spread option is an option whose pay-off depends on the price spread between two correlated underlying assets. If the asset prices are  $S_1$  and  $S_2$ , the pay-off to a spread option of strike  $K$  is  $[\omega(S_1 - S_2 - K)]^+$  where  $\omega = 1$  for a call and  $\omega = -1$  for a put. Early work on spread option pricing by Ravindran (1993) and Shimko (1994) assumed that each forward price process is a geometric Brownian motion with constant volatility and that these processes have a constant non-zero correlation: we label this the '2GBM' framework for short.

The 2GBM framework is tractable but it captures neither the implied volatility smiles that are derived from market prices of the vanilla options on  $S_1$  and  $S_2$ , nor the implied correlation smile that is evident from market prices of the spread options on  $S_1 - S_2$ . In fact, correlation 'frowns' rather than 'smiles' are a prominent feature in

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Matlab code that implements the closed form approximation described in this article is available upon request.

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spread option markets. This is because the pay-off to a spread option decreases with correlation. Since traders' expectations are usually of leptokurtic rather than normal returns, market prices of out-of-the-money call and put spread options are usually higher than the standard 2GBM model prices, which are based on a constant correlation. Hence, the implied correlations that are backed out from the 2GBM model usually have the appearance of a 'frown'.

Numerical approaches to pricing and hedging spread options that are both realistic and tractable include those of Carr and Madan (1999) and Dempster and Hong (2000) who advocate models that capture volatility skews on the two assets by introducing stochastic volatility to the price processes. And the addition of price jumps can explain the implied correlation frown, as in the spark spread option pricing model of Carmona and Durrleman (2003a). However, pricing and hedging in this framework necessitates computationally intensive numerical resolution methods such as the fast Fourier transform (see Hurd and Zhou, 2009). Other models provide only upper and lower bounds for spread option prices, as in Durrleman (2001) and Carmona and Durrleman (2005), who determine a price range that can be very narrow for certain parameter values. For a detailed survey of these models and a comparison of their performances, the reader is referred to the excellent survey by Carmona and Durrleman (2003b).

Spread option traders often prefer to use analytical approximations, rather than numerical techniques, for their computational ease and the availability of closed form formulae for hedge ratios. By reducing the dimension of the price uncertainty from two to one, the 2GBM assumption allows several quite simple closed form approximations for the spread option price to be derived (see Eydeland and Wolyniec, 2003). The most well known of these is the approximation stated by Kirk (1996), the exact origin of which is unknown; it is commonly referred to by traders as Kirk's approximation. Another approximation, due to Deng *et al.* (2008), is derived by expressing a spread option price as a sum of one-dimensional integrals, and Deng *et al.* (2007) extended this approximation, and Kirk's approximation, to price and hedge multi-asset spread options.

All these approximations are based on the 2GBM assumption, where the underlying prices are assumed to have a bivariate log-normal distribution, which is quite unrealistic for most types of financial assets. However, they may be extended to approximations for spread options under more general assumptions for the joint distribution of the underlying prices. For instance, Alexander and Scourse (2004) assumed that the underlying prices have a bivariate log-normal mixture distribution and hence expressed spread option prices as a weighted sum of four different 2GBM spread option prices, each of which may be obtained using an analytical approximation. The prices so derived display volatility smiles in the marginal distributions and a correlation frown in the joint distribution.

There is an indeterminacy problem with the 2GBM closed form approximations mentioned above. The problem arises because all these approximations assume we know the implied volatilities of the corresponding single-asset options. Since the 2GBM assumption is unrealistic, the individual asset implied volatilities are usually not constant with respect to the single-asset option strike – that is, there is, typically, an implied volatility skew for each asset. So the strike at which we measure the implied

volatility matters. However, this is not determined in the approximation. Thus, we have no alternative but to apply some ad hoc rule, which we call the *strike convention*. But the implied correlation is sensitive to the strike convention; that is, different rules for determining the single-asset option-implied volatilities give rise to quite different structures for the correlation frown. Indeed, many strike conventions give infeasible values for implied correlations when calibrating to market prices.

There are two sources of this problem. Firstly, some approximations (including Kirk's approximation) are only valid for spread options of certain strikes. Secondly, and this is due to the indeterminacy described above, correlation risk is not properly quantified in these approximations. In fact, the sensitivity of the spread option price to correlation is constrained to be directly proportional to the option vega. Indeed, a problem that is common to all approximations that require an ad hoc choice of strike convention is that the hedge ratios derived from such approximations may be inconsistent with the vanilla option Greeks, and as such, the errors from delta-gamma-vega hedging could be inappropriately attributed to correlation risk.

In this article we derive a new closed form approximation for spread option prices and hedge ratios, based on the 2GBM assumption. We express the spread option price as the sum of the prices of two compound exchange options (CEOs). One compound option is to exchange two vanilla call options, one on each of the two underlying assets, and the other compound option is to exchange the corresponding put options. In this CEO approximation, the strikes at which to measure the single-asset implied volatilities are endogenous to the model. Thus, the CEO approximation is free of any strike convention, and yields a unique implied correlation for each spread option strike, even when there are implied volatility skews on the individual assets. We shall demonstrate, using simulations of spread option prices, and using real market spread option prices, that the CEO approximation provides a very much closer fit to the spread option price than does Kirk's approximation. Moreover, the CEO hedge ratios are consistent with those for the single-asset options. Furthermore, correlation risk is not simply assumed to be proportional to volatility risk, as it is in other analytical approximations. In fact, correlation sensitivities are quantified independently of the single-asset option vegas. We also derive a new, general formula for the early exercise premium (EEP) of an American spread option on spot underlyings. This is because the majority of traded spread options are American-style options on assets that pay dividends or have costs associated with the convenience yield or hedge rollover.

The outline is as follows: Section 2 provides a critical review of the existing analytical approximations for spread options, exploring in greater depth the claims made above. Section 3 sets out the CEO representation and derives a new closed form approximation to the price and hedge ratios of European spread options. Section 4 derives the EEP for an American spread option. Section 5 reports the results of two empirical studies: it begins with a simulation exercise that demonstrates the flexibility and accuracy of the CEO approximation and explores the practical difficulties that arise on attempting to implement the approximation given by Kirk (1996). Then we calibrate the CEO approximation to market data for American crack spread options, comparing the fit and the spread option hedge ratios with those obtained using Kirk's approximation. Section 6 concludes.

## 2. Background

Let  $(\Theta, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{Q})$  be a filtered probability space, where  $\Theta$  is the set of all possible events  $\theta$  such that  $S_{1t}, S_{2t} \in (0, \infty)$ ,  $(\mathcal{F}_t)_{t \geq 0}$  is the filtration produced by the sigma algebra of the price pair  $(S_{1t}, S_{2t})_{t \geq 0}$  and  $\mathbb{Q}$  is a bivariate risk neutral probability measure. Assume that the risk neutral price dynamics are governed by two correlated geometric Brownian motions with constant volatilities, so the dynamics of the two underlying asset prices are given by

$$dS_{it} = (r - q_i)S_{it}dt + \sigma_i S_{it}dW_{it}, \quad i = 1, 2, \quad (1)$$

where  $W_{1t}$  and  $W_{2t}$  are Wiener processes under the risk neutral measure  $\mathbb{Q}$ ,  $r$  is the (assumed constant) risk-free interest rate and  $q_1$  and  $q_2$  are the (assumed constant) dividend or convenience yields of the two assets. The volatilities  $\sigma_1$  and  $\sigma_2$  are also assumed to be constant as is the covariance

$$\langle dW_{1t}, dW_{2t} \rangle = \rho dt.$$

When the strike of a spread option is zero the option is called an exchange option, since the buyer has the option to exchange one underlying asset for the other. The fact that the strike is zero allows one to reduce the pricing problem to a single dimension, using one of the assets as numeraire. If  $S_{1t}$  and  $S_{2t}$  are the spot prices of two assets at time  $t$  then the pay-off to an exchange option at the expiry date  $T$  is given by  $[S_{1T} - S_{2T}]^+$ . But this is equivalent to an ordinary call option on  $x_t = S_{1t}/S_{2t}$  with unit strike. Hence, using risk neutral valuation, the price of an exchange option is given by

$$P_t = \mathbb{E}_{\mathbb{Q}} \left\{ e^{-r(T-t)} [S_{1T} - S_{2T}]^+ \right\} = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left\{ S_{2T} [x_T - 1]^+ \right\}.$$

Margrabe (1978) showed that under these assumptions the price  $P_t$  of an exchange option is given by

$$P_t = S_{1t} e^{-q_1(T-t)} \Phi(d_1) - S_{2t} e^{-q_2(T-t)} \Phi(d_2), \quad (2)$$

where  $\Phi$  denotes the standard normal distribution function and

$$d_1 = \frac{\ln\left(\frac{S_{1t}}{S_{2t}}\right) + (q_2 - q_1 + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}};$$

$$d_2 = d_1 - \sigma\sqrt{T-t};$$

$$\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}.$$

A well-known approximation for pricing European spread options on futures or forwards, which is valid for small, non-zero strikes, appears to have been stated first by Kirk (1996). When  $K \ll S_{2t}$  the displaced diffusion process  $S_{2t} + K$  can be assumed to

be approximately log-normal, so the ratio between  $S_{1t}$  and  $(S_{2t} + Ke^{-r(T-t)})$  is also approximately log-normal and can be expressed as a geometric Brownian motion process. To see this rewrite the pay-off to the European spread option as

$$[\omega(S_{1T} - S_{2T} - K)]^+ = (K + S_{2T})[\omega(Z_T - 1)]^+,$$

where  $\omega = 1$  for a call and  $\omega = -1$  for a put,  $Z_t = \frac{S_{1t}}{Y_t}$  and  $Y_t = S_{2t} + Ke^{-r(T-t)}$ . Let  $W$  be a Brownian motion under a new probability measure  $\mathbb{P}$  whose Radon-Nikodym derivative with respect to  $\mathbb{Q}$  is given by

$$\frac{d\mathbb{P}}{d\mathbb{Q}} = \exp\left(-\frac{1}{2}\tilde{\sigma}_2^2 T + \tilde{\sigma}_2 W_{2t}\right).$$

Then  $Z$  follows a process described by

$$\frac{dZ_t}{Z_t} = (r - \tilde{r} - (q_1 - \tilde{q}_2)) dt + \sigma_t dW_t,$$

with

$$\sigma = \sqrt{\sigma_1^2 + \tilde{\sigma}_2^2 - 2\rho\sigma_1\tilde{\sigma}_2},$$

where  $\tilde{\sigma}_2 = \sigma_2 \frac{S_{2t}}{Y_t}$ ,  $\tilde{r} = r \frac{S_{2t}}{Y_t}$ ,  $\tilde{q}_2 = q_2 \frac{S_{2t}}{Y_t}$ . Therefore, the price  $f_t$  at time  $t$  for a spread option on  $S_1$  and  $S_2$  with strike  $K$ , maturity  $T$  and pay-off  $[\omega(S_1 - S_2 - K)]^+$  is given by

$$\begin{aligned} f_t &= \mathbb{E}_{\mathbb{Q}} \left\{ Y_t e^{-r(T-t)} [\omega(Z_T - 1)]^+ \right\} \\ &= \omega \left[ S_{1,t} e^{-q_1(T-t)} \Phi(\omega d_{1Z}) - \left( Ke^{-r(T-t)} + S_{2t} \right) e^{-(r - (\tilde{r} - \tilde{q}_2))(T-t)} \Phi(\omega d_{2Z}) \right], \end{aligned} \quad (3)$$

where

$$\begin{aligned} d_{1Z} &= \frac{\ln(Z_t) + (r - q_1 - (\tilde{r} - \tilde{q}_2) + \frac{1}{2}\sigma_t^2)(T-t)}{\sigma_t \sqrt{T-t}}, \\ d_{2Z} &= d_{1Z} - \sigma_t \sqrt{T-t}. \end{aligned}$$

Under Kirk's approximation, the spread option's deltas and gammas are given by

$$\begin{aligned} \Delta_{S_1}^f &= \omega e^{-q_1(T-t)} \Phi(\omega d_{1Z}), \\ \Delta_{S_2}^f &= -\omega e^{-q_1(T-t)} \Phi(\omega d_{2Z}), \end{aligned} \quad (4)$$

$$\Gamma_{S_1 S_1}^f = e^{-q_1(T-t)} \frac{\phi(d_{2Z})}{S_{1t} \sigma_t \sqrt{T-t}},$$

$$\Gamma_{S_2 S_2}^f = e^{-(r-\bar{r}+\bar{q}_2)(T-t)} \frac{\phi(d_{2Z})}{(Ke^{-r(T-t)} + S_{2t}) \sigma_t \sqrt{T-t}}.$$

The cross gamma, that is, the second-order derivative of price with respect to both the underlying assets is given by

$$\Gamma_{S_1 S_2}^f = -e^{-q_1(T-t)} \frac{\phi(d_{1Z})}{(Ke^{-r(T-t)} + S_{2t}) \sigma_t \sqrt{T-t}} = -e^{-(r-\bar{r}+\bar{q}_2)(T-t)} \frac{\phi(d_{2Z})}{S_{1t} \sigma_t \sqrt{T-t}}, \quad (5)$$

where  $\Delta_x^z$  denotes the delta of  $y$  with respect to  $x$  and  $\Gamma_{xy}^z$  denotes the gamma of  $z$  with respect to  $x$  and  $y$ . The Kirk-approximation vegas are similar to Black-Scholes vegas and are easy to derive using the chain rule.

Under the 2GBM assumption, other price approximations exist that also reduce the dimension of the uncertainty from two to one.<sup>1</sup> For instance, let  $S_t = S_{1t}e^{-q_1(T-t)} - S_{2t}e^{-q_2(T-t)}$  and choose an arbitrary  $M \gg \max\{S_t, \sigma_t\}$ . Then another spread option price, this time based on the approximation that  $M + S_t$  has a log-normal distribution, is

$$f_t = (M + S_t)\Phi(d_{1M}) - (M + K)e^{-r(T-t)}\Phi(d_{2M}),$$

where

$$d_{1M} = \frac{\ln\left(\frac{M+S_t}{M+K}\right) + \left(r - q + \frac{1}{2}\sigma_t^2\right)(T-t)}{\sigma_t \sqrt{T-t}};$$

$$d_{2M} = d_{1M} - \sigma_t \sqrt{T-t};$$

$$\sigma_t = \sqrt{\frac{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}{(M + S_t)}}.$$

We remark that, in most of the analytical approximations which are derived by reducing the price dimension from two to one, the spread option volatility will take the form

$$\sigma = \sqrt{\omega_1\sigma_1^2 + \omega_2\sigma_2^2 - 2\omega_3\rho\sigma_1\sigma_2}, \quad (6)$$

where the terms on the right-hand side are assumed to be constant.

To avoid arbitrage, a spread option must be priced consistently with the prices of vanilla options on  $S_1$  and  $S_2$ . This implies setting  $\sigma_t$  in Equation (6) is equal to the implied volatility of  $S_i$ , for  $i = 1, 2$ . Then the implied correlation is calibrated by equating the model and market prices of the spread option. But although the 2GBM model assumes constant volatility, the market implied volatilities are not constant with

respect to strike. So the strikes  $K_1$  and  $K_2$  at which the implied volatilities  $\sigma_1$  and  $\sigma_2$  are calculated can have a significant influence on the results. Each of the spread option price approximations reviewed above requires the single-asset implied volatilities to be determined by some convention for choosing  $(K_1, K_2)$  such that  $K_1 - K_2 = K$ . There are infinitely many possible choices for  $K_1$  and  $K_2$  and, likewise, infinitely many combinations of market implied  $\sigma_1$ ,  $\sigma_2$  and  $\rho$  that yield the same  $\sigma$  in Equation (3). Hence, the implied volatility and implied correlation parameters are ill-defined. Moreover, the sensitivity of volatility to correlation is constant, and this implies that the price sensitivity to correlation is directly proportional to the option vega (i.e. the partial derivative of the price w.r.t.  $\sigma$ ).

Spread options may be delta-gamma hedged by taking positions in the underlying assets and options on these. But hedging volatility and correlation may be much more difficult. Vega hedging is complicated by the fact that the hedge ratios depend on the strike convention. For a given strike  $K$ , there could be very many pairs  $(K_1, K_2)$  with  $K_1 - K_2 = K$  which provide an accurate price. However, for each such pair  $(K_1, K_2)$  the spread option Greeks will be different, and if the strikes are chosen without regard for vega risk, the hedging errors accruing from incorrect vega hedging, along with every other unhedged risk, will be collectively attributed to correlation risk.

For this reason we should impose a further condition in the strike convention, that is, that the spread option Greeks are consistent with the vanilla option Greeks. If they are not, there could be substantial hedging error from gamma and/or vega hedging the spread option with vanilla options. We define a *compatible* strike pair  $(K_1, K_2)$  with  $K_1 - K_2 = K$ , to be such that the price and the hedge ratios of the spread option with strike  $K$  are consistent with the prices and the hedge ratios of the two vanilla options at strikes  $K_1$  and  $K_2$ . A compatible pair  $(K_1, K_2)$  can be found by equating four ratios: two of the spread option deltas relative to the single-asset option deltas, and two of the spread option vegas relative to the single-asset option vegas. There is no condition for gamma because it is proportional to vega in the GBM framework.

Now, conditional on the two prices for the underlying assets at expiry, we can choose a unique pair of vanilla options on the respective assets that best reflects the market expectations of these prices at the option's expiry and which together replicates the spread option's pay-off.<sup>2</sup> Since the price of a spread option is a linear combination of the conditional expectation of the underlying asset prices, the implied volatilities of this unique pair of vanilla options are expected to give the most accurate spread option prices, assuming the market expectations are correct.

This hedging argument suggests that the strike convention should be selecting a compatible strike pair; this is one of the strike conventions that we have followed in this article. In addition, to explore whether more realistic implied correlations are obtained using non-compatible strike pairs, we have employed several other strike conventions: using the single asset's at-the-money (ATM) forward volatility to calibrate spread options of all strikes; several conventions for which each  $K_i$  is a linear function of  $K$  and  $S_i$ , for  $i = 1, 2$ ; and calibrating  $K_1$  as a model parameter, then setting  $K_2 = K_1 - K$ . However, as we shall see in Section 5, in no case did we obtain reasonable results for spread options of all strikes, when applied to either simulated or market data.

In the Section 3, we present a new closed form approximation in which a compatible pair of single-asset option strikes is endogenous. It is determined by calibrating the model to the vanilla option-implied volatility skews and to the implied correlation frown of the spread options of different strikes.

### 3. Compound Exchange Option Approach

In this section, we express the price of a spread option as a sum of prices of two CEOs, one on vanilla call options and the other on vanilla put options. The spread option pricing problem thus reduces to finding the right call option pair (and the right put option pair) and then calibrating the implied correlation between the two vanilla options. By establishing a conditional relationship between the strikes of vanilla options and the implied correlation, the spread option pricing problem reduces to a one-dimensional problem. We also derive a correlation sensitivity for the spread option price that is independent of the volatility hedge ratios.

*Theorem 3.1.* The risk neutral price of a European spread option may be expressed as the sum of risk neutral prices of two compounded exchange options. That is,

$$f_t = e^{-r(T-t)} (\mathbb{E}_{\mathbb{Q}} \{[\omega [U_{1T} - U_{2T}]]^+ | \mathcal{F}_t\} + \mathbb{E}_{\mathbb{Q}} \{[\omega [V_{2T} - V_{1T}]]^+ | \mathcal{F}_t\}), \quad (7)$$

where  $U_{1T}$  and  $V_{1T}$  are pay-offs to European call and put options on asset 1; and  $U_{2T}$  and  $V_{2T}$  are pay-offs to European call and put options on asset 2, respectively.

*Proof.* Let  $K_1$  and  $K_2$  be positive real numbers such that  $K_1 - K_2 = K$  and

$$\mathcal{L} = \{\theta \in \Theta : \omega (S_{1T} - S_{2T} - K) \geq 0\},$$

$$\mathcal{A} = \{\theta \in \Theta : S_{1T} - K_1 \geq 0\},$$

$$\mathcal{B} = \{\theta \in \Theta : S_{2T} - K_2 \geq 0\}.$$

Since a European option price at time  $t$  depends only on the terminal price densities, we have

$$\begin{aligned} f_t &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \{ \omega \mathbf{1}_{\mathcal{L}} [S_{1T} - S_{2T} - K] \} \\ &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \{ \omega \mathbf{1}_{\mathcal{L}} ( \mathbf{1}_{\mathcal{A}} [S_{1T} - K_1] - \mathbf{1}_{\mathcal{B}} [S_{2T} - K_2] \\ &\quad + (1 - \mathbf{1}_{\mathcal{A}}) [S_{1T} - K_1] - (1 - \mathbf{1}_{\mathcal{B}}) [S_{2T} - K_2] ) \} \\ &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \{ \omega ( \mathbf{1}_{\mathcal{L} \cap \mathcal{A}} [S_{1T} - K_1] - \mathbf{1}_{\mathcal{L} \cap \mathcal{B}} [S_{2T} - K_2] \\ &\quad + \mathbf{1}_{\mathcal{L}} (1 - \mathbf{1}_{\mathcal{B}}) [K_2 - S_{2T}] - \mathbf{1}_{\mathcal{L}} (1 - \mathbf{1}_{\mathcal{A}}) [K_1 - S_{1T}] ) \} \\ &= e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left\{ \left[ \omega \left( [S_{1T} - K_1]^+ - [S_{2T} - K_2]^+ \right) \right]^+ \right\} \\ &\quad + e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \left\{ \left[ \omega \left( [K_2 - S_{2T}]^+ - [K_1 - S_{1T}]^+ \right) \right]^+ \right\} \\ &= e^{-r(T-t)} (\mathbb{E}_{\mathbb{Q}} \{[\omega [U_{1T} - U_{2T}]]^+\} + \mathbb{E}_{\mathbb{Q}} \{[\omega [V_{2T} - V_{1T}]]^+\}), \end{aligned} \quad (8)$$

where  $U_{1T}$  and  $V_{1T}$  are pay-offs to European call and put options on asset 1 with strike  $K_1$ ; and  $U_{2T}$  and  $V_{2T}$  are pay-offs to European call and put options on asset 2 with strike  $K_2$ , respectively.  $\square$

The CEO representation of a spread option is a special case of the general framework for multi-asset option pricing introduced by Alexander and Venkatramanan (2011). Let  $U_{it}$  and  $V_{it}$  be the Black–Scholes option prices of the calls and puts in Equation (7), and set  $K_1 = mK$  to be the strike of  $U_1$  and  $V_1$  and  $K_2 = (m - 1)K$  to be the strike of  $U_2$  and  $V_2$ , for some real number  $m \geq 1$ . Choosing  $m$  so that the single-asset call options are deep in-the-money (ITM),<sup>3</sup> the risk neutral price at time  $t$  of a European spread option on 2GBM processes may be expressed as

$$f_t = e^{-r(T-t)} \omega \left[ U_{1t} \Phi(\omega d_{1U}) - U_{2t} \Phi(\omega d_{2U}) - (V_{1t} \Phi(-\omega d_{1V}) - V_{2t} \Phi(-\omega d_{2V})) \right], \quad (9)$$

where

$$d_{1A} = \frac{\ln\left(\frac{A_{1t}}{A_{2t}}\right) + (q_2 - q_1 + \frac{1}{2}\sigma_A^2)(T - t)}{\sigma_A \sqrt{T - t}}; \quad (10)$$

$$d_{2A} = d_{1A} - \sigma_A \sqrt{T - t};$$

and

$$\sigma_U = \sqrt{\xi_1^2 + \xi_2^2 - 2\rho\xi_1\xi_2}, \quad (11)$$

$$\sigma_V = \sqrt{\eta_1^2 + \eta_2^2 - 2\rho\eta_1\eta_2},$$

$$\xi_i = \sigma_i \frac{S_{it}}{U_{it}} \frac{\partial U_{it}}{\partial S_{it}},$$

$$\eta_i = \sigma_i \frac{S_{it}}{V_{it}} \left| \frac{\partial V_{it}}{\partial S_{it}} \right|.$$

The correlation  $\rho$  used to compute the exchange option volatility  $\sigma_U$  in Equation (11) is the implied correlation between the two vanilla calls (of strikes  $K_1$  and  $K_2$ ), which is the same as the implied correlation between the two vanilla puts, because the puts have the same strikes as the calls. And, since each vanilla option is driven by the same Wiener process as its underlying price, the implied correlation between the vanilla options is the implied correlation of the spread option with strike  $K = K_1 - K_2$ . As the deltas of the two vanilla options vary with their strikes, the implied correlation does too. For instance, if we fix  $K_1$  then, as the spread option strike increases,  $K_2$  decreases and the difference between the two deltas increases. Hence, the implied correlation will decrease as  $K$  increases, and increase as  $K$  decreases. In other words the correlation skew or frown becomes endogenous to the model.

*Proposition 3.2.* The spread option deltas, gammas and vegas of the price, refer to Equation (9), are given by

$$\begin{aligned}\Delta_{S_i}^f &= \Delta_{U_i}^f \Delta_{S_i}^{U_i} + \Delta_{V_i}^f \Delta_{S_i}^{V_i}, \\ \Gamma_{S_i S_i}^f &= \Gamma_{U_i}^f \left( \Delta_{S_i}^{U_i} \right)^2 + \Gamma_{S_i}^{U_i} \Delta_{U_i}^f + \Gamma_{V_i}^f \left( \Delta_{S_i}^{V_i} \right)^2 + \Gamma_{S_i}^{V_i} \Delta_{V_i}^f, \\ \Gamma_{S_1 S_2}^f &= \Gamma_{S_2 S_1}^f = \Gamma_{U_1 U_2}^f \Delta_{S_1}^{U_1} \Delta_{S_2}^{U_2} + \Gamma_{V_1 V_2}^f \Delta_{S_1}^{V_1} \Delta_{S_2}^{V_2}, \\ \mathcal{V}_{\sigma_i}^f &= \mathcal{V}_{\sigma_U}^f \frac{\partial \sigma_U}{\partial \sigma_i} + \mathcal{V}_{\sigma_V}^f \frac{\partial \sigma_V}{\partial \sigma_i} + \mathcal{V}_{\sigma_i}^{U_i} \Delta_{U_i}^f + \mathcal{V}_{\sigma_i}^{V_i} \Delta_{V_i}^f,\end{aligned}\tag{12}$$

where  $\Delta_x^z$  and  $\mathcal{V}_x^z$  denote the delta and vega of  $z$  with respect to  $x$ , respectively, and  $\Gamma_{xy}^z$  denotes the gamma of  $z$  with respect to  $x$  and  $y$ .

*Proof.* Differentiate Equation (9) using the chain rule.  $\square$

Equation (12) shows that the CEO model Greeks are functions of their respective single-asset option Greeks. Therefore, it is possible to construct a portfolio with single-asset call and put options to replicate the spread option. For instance, to hedge the price and volatility risk of a call spread option due to asset 1, we can buy  $(\Delta_{U_1}^f + \mathcal{V}_{\sigma_1}^{U_1})$  call options on asset 1 with price  $U_1$  and  $(\Delta_{V_1}^f + \mathcal{V}_{\sigma_1}^{V_1})$  put options on asset 1 with price  $V_1$ . Other risks can be hedged in a similar manner.<sup>4</sup>

A limitation of the analytical approximations reviewed in Section 2 is that correlation risk is not properly quantified: the spread option correlation sensitivity must be a constant times the option vega. By contrast, the CEO approximation yields a closed-form formula for the sensitivity of the spread option price to correlation. Write the approximate spread option price as  $f = f(U_1, U_2, V_1, V_2, \sigma_U, \sigma_V)$ . The approximation is structured so that the implied correlation is directly related to  $m$ , the only independent and therefore central parameter. The exchange option volatilities in Equation (11) are therefore also determined by  $m$ , so we may write

$$\frac{\partial f}{\partial \rho} = \frac{\partial f}{\partial m} \frac{\partial m}{\partial \rho},$$

where

$$\frac{\partial f}{\partial m} = \frac{\partial f}{\partial U_1} \frac{\partial U_1}{\partial m} + \frac{\partial f}{\partial U_2} \frac{\partial U_2}{\partial m} + \frac{\partial f}{\partial V_1} \frac{\partial V_1}{\partial m} + \frac{\partial f}{\partial V_2} \frac{\partial V_2}{\partial m} + \frac{\partial f}{\partial \sigma_U} \frac{\partial \sigma_U}{\partial m} + \frac{\partial f}{\partial \sigma_V} \frac{\partial \sigma_V}{\partial m}.$$

Now, unlike other analytical approximations, in the CEO approximation the volatility and correlation hedge ratios could be independent of each other, depending on our choice for  $m$ . The correlation affects the spread option price only through its effect on  $\sigma_U$  and  $\sigma_V$ , and we may choose  $m$  such that  $d\sigma_U/d\rho = d\sigma_V/d\rho = 0$ . We call the spread option volatility at such a value for  $m$  the *pure* spread option volatility. Thus, we may choose  $m$  so that

$$\frac{\partial f}{\partial m} = K \left( \frac{\partial f}{\partial U_1} \frac{\partial U_1}{\partial K_1} + \frac{\partial f}{\partial U_2} \frac{\partial U_2}{\partial K_2} + \frac{\partial f}{\partial V_1} \frac{\partial V_1}{\partial K_1} + \frac{\partial f}{\partial V_2} \frac{\partial V_2}{\partial K_2} \right),$$

in which case, at the pure spread option volatility, we have

$$\frac{\partial f}{\partial \rho} = Kg(\xi_1, \xi_2, \rho; m)^{-1} \left( \frac{\partial f}{\partial U_1} \frac{\partial U_1}{\partial K_1} + \frac{\partial f}{\partial U_2} \frac{\partial U_2}{\partial K_2} + \frac{\partial f}{\partial V_1} \frac{\partial V_1}{\partial K_1} + \frac{\partial f}{\partial V_2} \frac{\partial V_2}{\partial K_2} \right), \quad (13)$$

where

$$g(x, y, z, m) = (xy)^{-1} \left( \left( \frac{\partial x}{\partial m} x + \frac{\partial y}{\partial m} y \right) - z \left( \frac{\partial x}{\partial m} y + \frac{\partial y}{\partial m} x \right) \right).$$

We call Equation (13) the *pure* correlation sensitivity of the spread option price because it is independent of the volatility sensitivities  $\partial f/\partial\sigma_U$  and  $\partial f/\partial\sigma_V$ . That is, the pure correlation sensitivity of the spread option price is the correlation sensitivity at the pure spread option volatility. In the following proposition, we establish a precise relationship between the pure correlation sensitivity of the spread option price and our central parameter  $m$ :

*Proposition 3.3.* Let  $m = m(\rho, t)$  be such that  $m : [-1, 1] \times [0, T] \rightarrow [1, \infty)$ . Then at the pure spread option volatility, we have

$$g(\xi_1, \xi_2, \rho; m) = g(\eta_1, \eta_2, \rho; m). \quad (14)$$

*Proof.* The total derivative of  $\sigma_U$  is

$$d\sigma_U = \frac{\partial\sigma_U}{\partial\xi_1} d\xi_1 + \frac{\partial\sigma_U}{\partial\xi_2} d\xi_2 + \frac{\partial\sigma_U}{\partial\rho} d\rho.$$

Hence,

$$\begin{aligned} \frac{d\sigma_U}{d\rho} &= \frac{\partial\sigma_U}{\partial\xi_1} \frac{d\xi_1}{dm} \frac{dm}{d\rho} + \frac{\partial\sigma_U}{\partial\xi_2} \frac{d\xi_2}{dm} \frac{dm}{d\rho} + \frac{\partial\sigma_U}{\partial\rho} \\ &= \mathbb{A} \frac{dm}{d\rho} - \frac{\xi_1 \xi_2}{\sigma_U}, \end{aligned} \quad (15)$$

where

$$\mathbb{A} = \sigma_U^{-1} \left( \frac{d\xi_1}{dm} (\xi_1 - \rho\xi_2) + \frac{d\xi_2}{dm} (\xi_2 - \rho\xi_1) \right).$$

Similarly

$$\frac{d\sigma_V}{d\rho} = \mathbb{B} \frac{dm}{d\rho} - \frac{\eta_1 \eta_2}{\sigma_V}, \quad (16)$$

where

$$\mathbb{B} = \sigma_V^{-1} \left( \frac{d\eta_1}{dm} (\eta_1 - \rho\eta_2) + \frac{d\eta_2}{dm} (\eta_2 - \rho\eta_1) \right).$$

When  $d\sigma_U/d\rho = 0$ , Equation (15) implies that

$$\begin{aligned} \frac{dm}{d\rho} &= \mathbb{A}^{-1} \frac{\xi_1 \xi_2}{\sigma_U} \\ &= \xi_1 \xi_2 \left( \frac{d\xi_1}{dm} (\xi_1 - \rho\xi_2) + \frac{d\xi_2}{dm} (\xi_2 - \rho\xi_1) \right)^{-1} \\ &= \xi_1 \xi_2 \left( \left( \frac{d\xi_1}{dm} \xi_1 + \frac{d\xi_2}{dm} \xi_2 \right) - \rho \left( \frac{d\xi_1}{dm} \xi_2 + \frac{d\xi_2}{dm} \xi_1 \right) \right)^{-1} \\ &= g(\xi_1, \xi_2, \rho; m)^{-1}. \end{aligned}$$

Similarly, when  $d\sigma_V/d\rho = 0$ ,

$$\begin{aligned} \frac{dm}{d\rho} &= \eta_1 \eta_2 \left( \left( \frac{d\eta_1}{dm} \eta_1 + \frac{d\eta_2}{dm} \eta_2 \right) - \rho \left( \frac{d\eta_1}{dm} \eta_2 + \frac{d\eta_2}{dm} \eta_1 \right) \right)^{-1} \\ &= g(\eta_1, \eta_2, \rho; m)^{-1}. \end{aligned}$$

Therefore,  $g(\xi_1, \xi_2, \rho; m) = g(\eta_1, \eta_2, \rho; m)$ . Finally, note that  $dm/d\rho$  is well-defined throughout  $\rho \in [-1, 1]$  because if

$$\rho = \left( \frac{d\xi_1}{dm} \xi_1 + \frac{d\xi_2}{dm} \xi_2 \right) \left( \frac{d\xi_1}{dm} \xi_2 + \frac{d\xi_2}{dm} \xi_1 \right)^{-1},$$

then  $g(\xi_1, \xi_2, \rho; m) = 0$  and  $g(\eta_1, \eta_2, \rho; m) = 0$  if  $\xi_i = \eta_i$  or  $\eta_i = 0$ . But  $\xi_i$  can never be equal to  $\eta_i$ , and when  $\eta_i = 0$ , the spread option is replicated by the CEO on calls (there is no CEO on puts) and we do not need Equation (14). Therefore,

$$\rho \neq \left( \frac{d\eta_1}{dm} \eta_1 + \frac{d\eta_2}{dm} \eta_2 \right) \left( \frac{d\eta_1}{dm} \eta_2 + \frac{d\eta_2}{dm} \eta_1 \right)^{-1}.$$

Proposition 3.2 provides a condition (i.e.  $g(\xi_1, \xi_2, \rho; m) = g(\eta_1, \eta_2, \rho; m)$ ) that we shall use to calibrate the CEO approximation at the pure spread option volatility. This way we obtain a correlation sensitivity for the spread option price that is not constrained to be directly proportional to its volatility sensitivity. At time  $t$  we calibrate a single parameter  $m = m(\rho, t)$  for each spread option, by equating the market price of the spread option to its model price (Equation (9)). Let  $f_M$  be the market price of the spread option and  $f_i(m, \rho)$  be the price of a spread option given by Equation (9). Then, for this option, we choose  $m$  such that  $\|f_M - f(m, \rho)\|$  is minimized, subject to the

constraint that  $g(\xi_1, \xi_2, \rho; m_j) = g(\eta_1, \eta_2, \rho; m_j)$  at each iteration  $j$ .<sup>5</sup> Then, a compatible pair of single-asset options' strikes is uniquely determined by setting  $K_1 = mK$  and  $K_2 = (m - 1)K$ , where  $K$  is the strike of the spread option.

The calibration problem can be solved using a one-dimensional gradient method or, for faster converge, a one-dimensional quasi-Newton method. The first-order differential of  $f$  with respect to  $\rho$  is given by Equation (13). The first-order derivatives of  $\xi_i$  and  $\eta_i$  with respect to  $m$  can be calculated from their respective implied volatilities  $\sigma_1$  and  $\sigma_2$  either numerically or by assuming a parametric function, such as a cubic spline, on their strikes. Then  $m$  can be calculated either numerically, or by finding the roots of the resulting polynomial equation.<sup>6</sup> Therefore, the model calibration will just involve using a one-dimensional solver method, so the computation time will be minimal.

#### 4. Pricing American Spread Options

The price of an American-style option on a single underlying asset is mainly determined by the type of the underlying asset, the prevailing discount rate and the presence of any dividend or convenience yield. The option to exercise early suggests that these options are more expensive than their European counterparts but there are many instances when early exercise is not optimal, for instance for calls on non-dividend paying stocks, and calls or puts on forward contracts (see James, 2003). Since no traded options are perpetual, the expiry date forces the price of American options to converge to the price of their European counterparts. Before expiry, the prices of American calls and puts are always greater than or equal to the corresponding European calls and puts.

Even for the case of a single asset, there are no closed form solutions to the price of an American option (except for few cases like a perpetual American call option or an American call option on an asset that pays no dividend in which case the solution is trivial). Existing work on pricing American options ranges from analytical approximations through semi-analytical approaches (which may require some numerical computation) to full numerical schemes. In the case of American spread options, the pricing problem gets more complicated due to an increase in dimensionality and the non-existence of an analytical solution even to the price of a European spread option.

In this article, as a natural extension to our closed form approximation, we adopt the EEP integral equation approach which is semi-analytical in nature. In the EEP integral equation approach, the price of an American option is expressed as a sum of its European counterpart and an EEP that is computed as an integral.<sup>7</sup> Here the optimal stopping problem is transformed to one that involves finding the boundary point at which it is optimal to exercise. The EEP is then the expected value of the net gains from the pay-off from early exercise, conditional on the underlying asset price crossing the optimal boundary.

For instance, the price of an American call option with strike  $K$  on one underlying asset with price process (1) is given by

$$P(S_{1t}, t) = P^E(S_{1t}, K, r, q, \sigma, T) + \int_0^\tau q S_{1t} e^{-q(s-t)} \Phi(d_1(S_{1t}, B_{1t}, s-t)) ds - \int_0^\tau r K e^{-r(s-t)} \Phi(d_2(S_{1t}, B_{1t}, s-t)) ds,$$

where  $B_{1t}$  is the early exercise boundary. In order to compute the price of an American option using the above equation, we first need to solve for the exercise boundary. By noting that the price of an American option at maturity is equal to its pay-off, one may reduce the above equation to the much simpler form:

$$B_{1t} - K = P^E(B_{1t}, K, r, q, \sigma, T) + \int_0^\tau q S_{1t} e^{-q(s-t)} \Phi(d_1(S_{1t}, B_{1t}, s-t)) ds \\ - \int_0^\tau r K e^{-r(s-t)} \Phi(d_2(S_{1t}, B_{1t}, s-t)) ds,$$

which is sometimes referred to as value-match condition. However, since the exercise boundary is also a function of time, the integral on the right-hand side of the equation makes the solution pathdependent on the exercise boundary.

Many solutions have been proposed that either impose restrictions on the exercise boundary or assume it to have certain functional form. For instance, Kim (1990) proposed a solution to compute the price of a single-asset American option where he approximates the exercise boundary using a step function across time. This is achieved by discretizing the time interval into  $n$  subintervals and solving an integral equations to compute the exercise boundary for each subinterval  $[0, t_i]$  where  $\bigcup_1^{n-1} [t_i, t_{i+1}] \subseteq [0, \tau]$  and  $t_0 = 0$ . This is done as follows: (1) the exercise boundary is assumed to remain constant within an interval  $[0, t_i]$ , which removes the time dependency of the exercise boundary for that interval; (2) the discretized value-match equation will then reduce to a fixed point problem of the form  $x = f(x)$ , and one may be able to apply an iterative scheme to compute the exercise boundary; and (3) once the exercise boundaries are computed for each subinterval  $[0, t_i]$  for  $i = 1, 2, \dots, n$ , the boundaries are pieced together to give a stepwise function which is substituted back into the price equation to compute the American option price.<sup>8</sup>

We now extend our pricing approach to derive a semi-analytical approximate solution to an American spread option price. First, we may express this price in the form

$$P_t^A(S_1, S_2, K, q_1, q_2, \sigma, T) = \sup_{\tau \in \mathcal{S}_{t,T}} \mathbb{E}_{\mathbb{Q}} \left\{ e^{-r(\tau-t)} Y_\tau | \mathcal{F}_t \right\}, \quad (17)$$

where  $Y_t = \omega [S_{1t} - S_{2t} - K]^+$  is the spread option pay-off at time  $t$ ;  $\mathcal{S}_{t,T}$  denotes the set of all stopping times between  $t$  and  $T$ ; and the reason for the option price dependence on  $q_1$  and  $q_2$  will be clarified below. Next, applying Tanaka–Meyer’s formula (see Karatzas and Shreve, 1991) to the pay-off, we may write

$$Y_t = Y_0 + A_t^Y + M_t^Y,$$

where  $M^Y$  is a  $\mathbb{Q}$ -martingale and  $A^Y$  is a difference of non-decreasing processes null at 0, adapted to the filtration  $(\mathcal{F})_{t \geq 0}$ . Now the value of an American spread option at  $t \in [0, \tau_0]$  can be expressed as

$$P_t^A(S_1, S_2, K, q_1, q_2, \sigma, T) = e^{-r(T-t)} \mathbb{E}_{\mathbb{Q}} \{ Y_T \} + \mathbb{E}_{\mathbb{Q}} \left\{ \int_{\tau_0}^T e^{-r(s-\tau_s)} \mathbf{1}_{\tau_s=s} (r Y_s ds - dA_s^Y) \right\}, \quad (18)$$

where  $\tau_t = \inf \left\{ s \in [t, T] : Y_s = \sup_{\tau \in S_{s,T}} \mathbb{E}_{\mathbb{Q}} \{ e^{-r(\tau-t)} Y_\tau \} \right\}$  and

$$dA_t^Y = \omega ((r - q_1) S_{1t} dt - (r - q_2) S_{2t} dt).$$

Therefore, in the exercise region, where the spread option is ITM,

$$\begin{aligned} r Y_s ds - dA_s^Y &= r \omega (S_{1s} - S_{2s} - K) ds - \omega ((r - q_1) S_{1s} - (r - q_2) S_{2s}) ds \\ &= \omega (q_1 S_{1s} - q_2 S_{2s} - rK) ds. \end{aligned}$$

Setting  $\omega = 1$  and rewriting Equation (18) in terms of the moving boundary  $B_s = B(S_2, s)$  for  $s \in [0, \tau]$  and  $\tau = T - t$ , we obtain the price of a call spread option as

$$\begin{aligned} P_t^A(S_1, S_2, K, q_1, q_2, \sigma, T) &= P_t^E(S_1, S_2, K, q_1, q_2, \sigma, T) \\ &\quad + \mathbb{E}_{\mathbb{Q}} \left\{ \int_0^\tau e^{-r(s-t)} \mathbf{1}_{\{S_{1s} \geq B_s\}} (q_1 S_{1s} - q_2 S_{2s} - rK) ds \right\}. \end{aligned}$$

Before we can compute the price of the American spread option using the above equation, we must compute the exercise boundary  $B_t$ , which was earlier obtained by solving the value-match equation. Although we may be able to derive the value-match condition corresponding to the above equation, solving that for the exercise boundary could be computationally intensive because the expectation must be computed as a double integral over a bivariate density function, which further needs to be integrated over time. Thus, even if we were to adopt an iterative scheme to calculate the boundary for each time step, as in Kim (1990), a triple integral has to be evaluated for every iteration involved in computing the exercise boundary for each time step.

However, our CEO approximation allows one to tackle this problem differently. We start by noting that there is an alternative formulation for the CEO spread option price in Equation (9) that may be used to derive an approximate price of an American spread option. This is

$$f_t = S_{1t} e^{-q_1(T-t)} \mathbb{P}_1 - S_{2t} e^{-q_2(T-t)} \mathbb{P}_2 - K e^{-r(T-t)} \mathbb{P}_3,$$

where

$$\begin{aligned} \mathbb{P}_1 &= \mathbb{P}_1(S_{1t}, S_{2t}, K, q_1, q_2, \sigma, t) = \mathbb{P}(S_{1T} \geq S_{2T} + K) \\ &= \Phi(d_{11}) \Phi(\omega d_{1U}) - \Phi(-d_{11}) \Phi(-\omega d_{1V}), \\ \mathbb{P}_2 &= \mathbb{P}_2(S_{1t}, S_{2t}, K, q_1, q_2, \sigma, t) = \mathbb{P}(S_{2T} \leq S_{1T} - K) \\ &= \Phi(-d_{12}) \Phi(-\omega d_{2V}) - \Phi(d_{12}) \Phi(\omega d_{2U}), \end{aligned} \quad (19)$$

$$\begin{aligned}
\mathbb{P}_3 &= \mathbb{P}_3(S_{1t}, S_{2t}, K, q_1, q_2, \sigma, t) = \mathbb{P}(K \leq S_{1T} - S_{2T}) \\
&= \omega [(m-1)(\Phi(d_{22})\Phi(\omega d_{2U}) + \Phi(-\omega d_{2V})\Phi(-d_{22})) \\
&\quad - m(\Phi(d_{21})\Phi(\omega d_{1U}) + \Phi(-\omega d_{1V})\Phi(-d_{21}))].
\end{aligned}$$

Note that  $\mathbb{P}_1$ ,  $\mathbb{P}_2$  and  $\mathbb{P}_3$  are just the ITM conditional probabilities when the numeraire is each of the underlying asset prices and the bond price, respectively. Using the approximate CEO conditioning probabilities for the spread option to be ITM, we may express the EEP as a sum of three components, each of which is a one-dimensional integral over time of the probabilities shown above. Since these probabilities are in turn functions of univariate cumulative distribution functions, the overall computation time required for evaluating Equation (20) once will be of order  $O(n^2)$  instead of  $O(n^3)$ . This allows us to treat the American spread option problem as a simple extension of the single-asset American option price problem. As a result, we may use any of the existing approaches to price single-asset American options, such as Kim (1990), to compute the optimal boundaries.

Using the CEO probabilities in Equation (19), the American call spread option price can be expressed as

$$\begin{aligned}
P_t^A(S_1, S_2, K, q_1, q_2, \sigma, T) &= P_t^E(S_1, S_2, K, q_1, q_2, \sigma, T) \\
&\quad + \int_0^\tau q_1 S_{1t} e^{-q_1(s-t)} \mathbb{P}_1(S_{1t}, B_s, K, q_1, q_2, \sigma, s-t) ds \\
&\quad - \int_0^\tau q_2 S_{2t} e^{-q_2(s-t)} \mathbb{P}_2(S_{1t}, B_s, K, q_1, q_2, \sigma, s-t) ds \\
&\quad - \int_0^\tau r K e^{-r(s-t)} \mathbb{P}_3(S_{1t}, B_s, K, q_1, q_2, \sigma, s-t) ds.
\end{aligned} \tag{20}$$

The exercise boundary  $B(S_{2t}, t)$  solves the value-match equation,

$$\begin{aligned}
B(S_{2t}, t) - K &= P_t^E(S_{1t}, B_t, K, q_1, q_2, \sigma, T) \\
&\quad + \int_0^\tau q_1 B_t e^{-q_1(s-t)} \mathbb{P}_1(B_s, S_{2t}, K, q_1, q_2, \sigma, s-t) ds \\
&\quad - \int_0^\tau q_2 S_{2t} e^{-q_2(s-t)} \mathbb{P}_2(B_s, S_{2t}, K, q_1, q_2, \sigma, s-t) ds \\
&\quad - \int_0^\tau r K e^{-r(s-t)} \mathbb{P}_3(B_s, S_{2t}, K, q_1, q_2, \sigma, s-t) ds,
\end{aligned} \tag{21}$$

subject to boundary conditions:

$$B(S_{2t}, T) = \max \left\{ \frac{q_2}{q_1} S_{2t} + \frac{r}{q_1} K, S_{2t} + K \right\}, \quad (22)$$

$$B(0, t) = B_1(t),$$

where  $B_1(t)$  is the exercise boundary of a single-asset American option on asset 1 with strike  $K$ .

In the case of a two-asset American option, the early exercise boundary is a surface instead of a curve. However, using the approach of Kim (1990), we only need to compute the exercise boundary for a particular value of  $S_2$ , so we divide the time interval into  $n$  subintervals and rewrite the value-match condition as

$$B(S_{2t}, t_i) - K = P_t^E(S_{1t}, B_{t_i}, K, q_1, q_2, \sigma, T)$$

$$+ \int_0^{t_i} q_1 B_{t_i} e^{-q_1(s-t)} \mathbb{P}_1(B_s, S_{2t}, K, q_1, q_2, \sigma, s-t) ds$$

$$- \int_0^{t_i} q_2 S_{2t} e^{-q_2(s-t)} \mathbb{P}_2(B_s, S_{2t}, K, q_1, q_2, \sigma, s-t) ds \quad (23)$$

$$- \int_0^{t_i} r K e^{-r(s-t)} \mathbb{P}_3(B_s, S_{2t}, K, q_1, q_2, \sigma, s-t) ds.$$

Solving the above equation for every timestep will give us a stepwise approximation for the exercise boundary, which can be substituted back into Equation (20) to compute the price of an American spread option.

## 5. Empirical Results

We begin by calibrating the CEO approximation to simulated spread option prices and comparing the calibration errors with those derived from Kirk's approximation. For the simulations we have used prices  $S_1 = 65$  and  $S_2 = 50$ , and spread option strikes ranging between 9.5 and 27.5 with a step size of 1.5 and maturity of 30 days. To simulate market prices with implied volatility skews and a correlation frown, we used quadratic local volatility and local correlation functions that are assumed to be functionally dependent only on the price levels of the underlying assets and not on time. The dividend yields on both underlying assets are zero, the ATM volatilities were both 30% and the ATM correlation was 0.80.

In Kirk's approximation we set the strike convention and hence fix the single-asset implied volatilities. Then we use an iterative method to back-out the implied correlation for each option by setting Kirk's price equal to the simulated price. When we match the Kirk prices to our simulated market prices, it is very often impossible to derive a feasible value for the implied volatility, and/or for the implied correlation of the spread option in Kirk's formula. Instead we must constrain both these parameters to lie within their feasible set, and because of this there may be large differences between the Kirk's price and the market price.

Of the several strike conventions considered, the one that produced the smallest pricing errors (with both simulated and market data) was

$$K_1 = S_{1,0} - \frac{K}{2}, \quad K_2 = S_{2,0} + \frac{K}{2}.$$

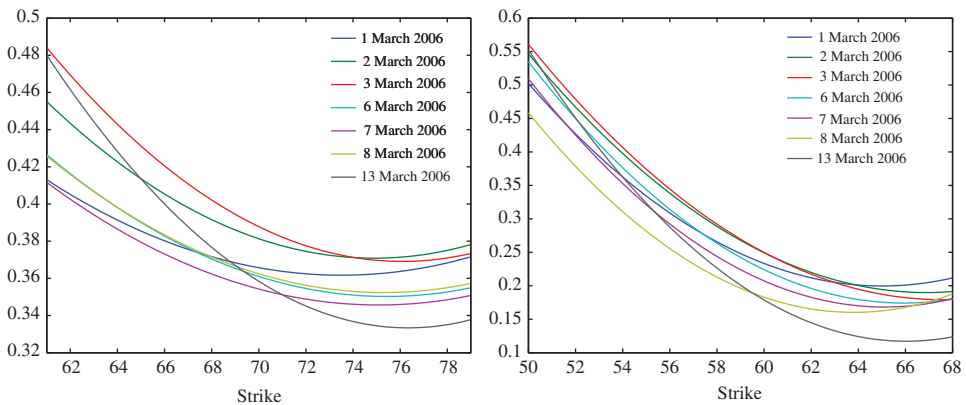
Still, in our simulated data, the root mean square calibration error (RMSE) was very high, at 9%.<sup>9</sup> By contrast, the CEO approximation's pricing errors are extremely small: the RMSE was 0.1% on the simulated data. Moreover, the CEO-implied correlation skews showed much greater stability over different simulations than those obtained using the Kirk's approximation, with any of the strike conventions.

This simulation exercise illustrates a major problem with the approximations for spread options surveyed in Section 2. That is, we have to apply a convention for fixing the strikes of the implied volatilities  $\sigma_1$  and  $\sigma_2$ , take the implied volatilities from the single-asset option prices and then calibrate the implied correlation to the spread option price. Moreover, using Kirk's approximation for illustration, we obtained unrealistic results whatever the strike convention employed, because for high strike spread options the model's log-normality assumption is not valid.

We now test the pricing performance of the CEO approximation using market prices of the 1:1 American put crack spread options that were traded on NYMEX between September 2005 and May 2006. These options are on the gasoline–crude oil spread and are traded on the price differential between the futures contracts of WTI light sweet crude oil and gasoline. Option data for American style contracts on each of these individual futures contracts were also obtained for the same time period, along with the futures prices. The size of all the futures contracts is 1000 bbls.

Figure 1 depicts the implied volatility skews in gasoline and crude oil on several days in March 2006, these being days with particularly high trading volumes. Pronounced negative implied volatility skews are evident in this figure, indicating that a suitable pricing model should be able to capture a skewed implied correlation frown.

Table 1 compares the results of Kirk's approximation with the CEO approximation by reporting the average absolute and percentage pricing errors on spread options with



**Figure 1.** Implied volatility of gasoline (left) and crude oil (right).

**Table 1.** Average absolute (percentage) pricing errors.

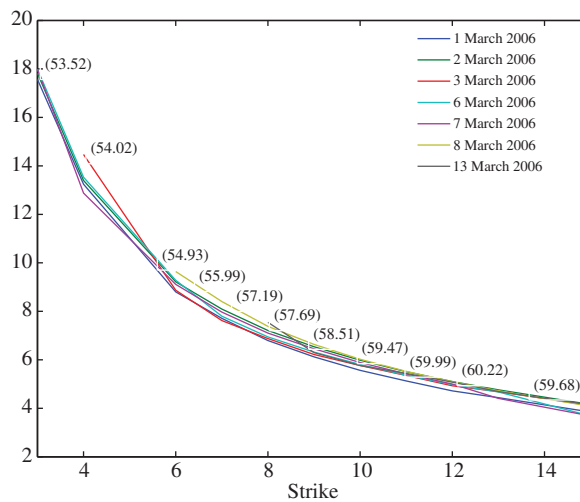
Strike	4.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0	13.0	15.0
CEO	0.0017 (0.64%)	0.0059 (0.69%)	0.0046 (0.68%)	0.0048 (0.47%)	0.0038 (1.01%)	0.0000 (0.001%)	0.0001 (0.002%)	0.0082 (0.24%)	0.0146 (0.55%)	0.0003 (0.005%)
Kirk's	0.4969 (179.1%)	0.6941 (122.7%)	0.7585 (176.7%)	0.8422 (120.1%)	0.8998 (84.7%)	0.92396 (60.5%)	0.9158 (44.1%)	0.8802 (32.2%)	0.8696 (26.1%)	0.8236 (13.2%)

different strikes, where the average is taken over all consecutive trading days between 1 and 15 March 2006. The models were calibrated to the market prices of both the gasoline–crude oil crack spread options and the individual gasoline and crude oil options.

Using Kirk’s approximation led to exactly the same calibration problems as were encountered with our simulated data. For high strikes, Kirk’s approximation based on feasible values for the implied correlation (between  $-1$  and  $+1$ ) gave prices that were far too low, and the opposite was the case with the low strikes. Only for a few strikes in the mid range were feasible values of the implied correlation found without constraining the iteration. Kirk’s approximation, with constrained values for the spread option’s implied volatility and correlation, gives an error that increases drastically for high strike values, as was also the case in our simulation results. By contrast, the CEO approximation errors were again found to be close to zero for all strikes on all dates.

Figure 2 plots the CEO parameter  $m$  as a function of the spread option strike, for the same days as in Figure 1.<sup>10</sup> Notice that, even though the implied volatilities shown in Figure 1 are quite variable from day to day,  $m$  is very stable at all strikes. The stability of  $m$  allows us to choose accurate starting values for calibration, and this reduces the calibration time to just a few seconds on a standard PC. Figure 3 shows that the implied correlations that are calibrated from the CEO approximation exhibit a realistic, negatively sloped skew on each day of the sample.

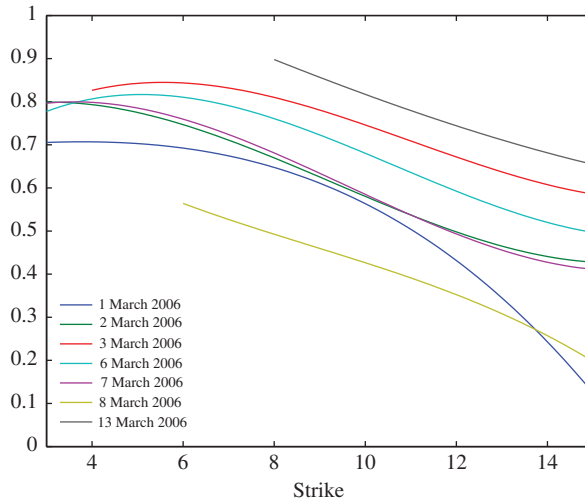
Table 2 compares the two deltas and the two gammas of each model, averaged over the sample period from 1 to 15 March 2006, as a function of the spread option strike. For low strikes the Kirk’s deltas are greater than the CEO deltas, and the opposite is the case at high strikes. On the other hand, the CEO gammas are higher than Kirk’s at most strikes. This demonstrates that, in addition to serious mispricing, the use of Kirk’s approximation will lead to inaccurate hedging, as was claimed in Section 2.



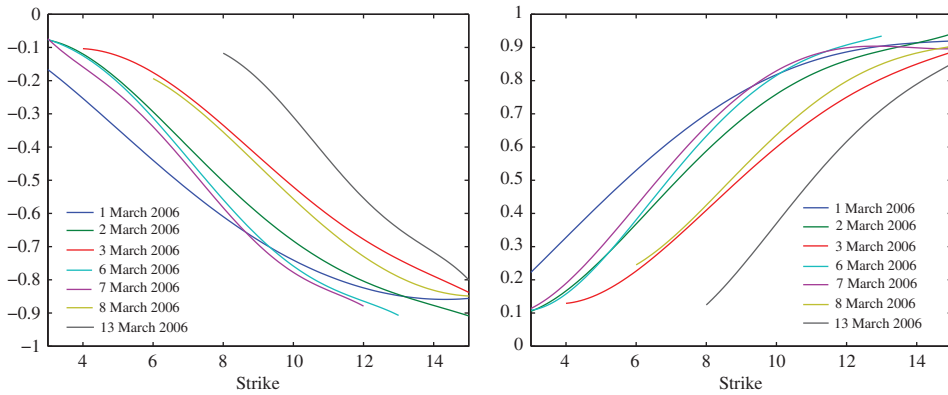
**Figure 2.** CEO parameter  $m$  with average strike of gasoline options ( $K_1 = mK$ ).

**Table 2.** Average difference between Kirk's and CEO deltas and gammas (put spread options).

Strike	3.0	4.0	6.0	7.0	8.0	9.0	10.0	11.0	12.0	13.0	15.0
$\Delta_{s_1}$	0.0405	0.0467	0.0462	0.0486	0.0463	0.0112	-0.0186	-0.0455	-0.0652	-0.0669	-0.0603
$\Delta_{s_2}$	0.0331	0.0338	0.0206	0.0180	0.0160	-0.0248	-0.0549	-0.0786	-0.0928	-0.1006	-0.0828
$\Gamma_{s_1}$	0.0091	-0.0010	-0.0364	-0.0492	-0.0478	-0.0519	-0.0522	-0.0469	-0.0375	-0.0234	-0.0080
$\Gamma_{s_2}$	0.0136	-0.0002	-0.0547	-0.0623	-0.0510	-0.0509	-0.0576	-0.0541	-0.0389	-0.0111	0.0162



**Figure 3.** Implied correlation skews of CEO approximation.



**Figure 4.** CEO delta with respect to gasoline (left) and crude oil (right).

Figures 4–6 depict the two deltas, gammas and vegas of the CEO approximation, respectively, as a function of the spread option strike.<sup>11</sup> At the ATM strikes the absolute deltas are close to 0.5 and the gammas are close to their maximum value. The vegas are greatest at ITM strikes, and their values depend on the implied volatility levels of the individual vanilla options used. In our data, the implied volatilities of gasoline were higher than those of crude oil in general, hence the peak of the vega with respect to gasoline is further to the right than the peak of the crude oil vega. Moreover, at every strike, vega varies considerably from day to day, because its value depends on the level of the spread and on the level of the respective underlying asset price. This is in sharp contrast to the hedge ratios derived using Kirk’s approximation, which are only affected by the level of the spread and not by the level of the underlying asset prices. Finally, Figure 7 depicts the CEO vega with respect to the pure spread option volatility.

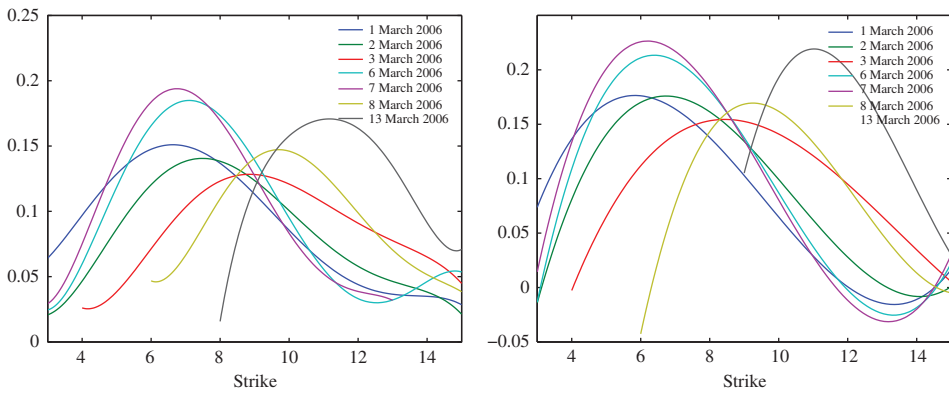


Figure 5. CEO gamma with respect to gasoline (left) and crude oil (right).

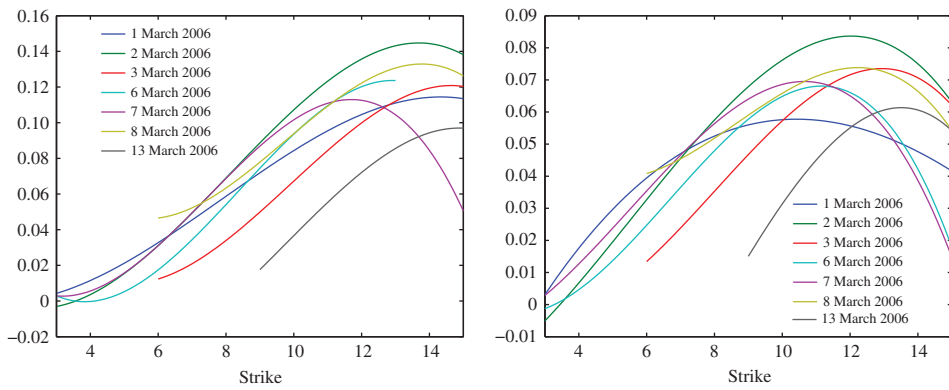


Figure 6. CEO vega with respect to gasoline (left) and crude oil (right).

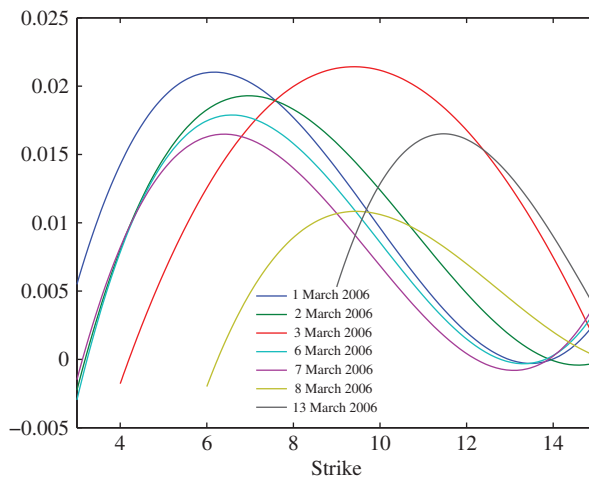


Figure 7. CEO vega with respect to spread option volatility.

Its strike dependence is quite similar to that of a Black–Scholes vanilla option vega, in that it takes its maximum value close to the ATM strike.

## 6. Conclusion

This article begins by highlighting the difficulties encountered when attempting to price and hedge spread options using closed form approximations based on a reduction of the price dimension. Firstly, in the presence of market-implied volatility smiles for the two underlyings, an arbitrary strike convention is necessary, and since the approximate prices and hedge ratios depend on this convention, they are not unique. As a result, the implied correlations that are implicit in the approximation may vary considerably, depending on the choice of strike convention. Secondly, the approximation may only be valid for a limited strike range. Thirdly, since the spread option prices are affected only by the relative price levels of the underlying assets and not by their individual levels, the probability that the spread option expires ITM is not tied to the price level. Thus, when equating the price approximation to a market price, we may obtain infeasible values for the implied volatility and/or correlation of the spread option. Fourthly, depending on the strike convention used, the hedge ratios derived from such approximations may be inconsistent with the single-asset option Greeks. And finally, the correlation sensitivity of the spread option price is simply assumed to be proportional to the option's vega.

We have developed a new closed form approximation based on an exact representation of a European spread option price as the sum of the prices of two CEOs and have derived an extension of this approximation to American spread options. Using both market and simulated data, we have demonstrated that our approximation provides accurate prices and realistic, unique values for implied correlation at all strikes. Another feature of our approximation that is not shared by other approximations is that the spread option Greeks are consistent with the single-asset option Greeks. This should lead to more accurate delta-gamma-vega hedging of spread option positions, using the two underlyings and vanilla calls and puts with strikes that are calibrated in the approximation.

## Acknowledgement

We thank Prof. Thorsten Schmidt of the Department of Mathematics, Leipzig University, and Andreas Kaeck, ICMA Centre, for very useful comments on a earlier draft of this article.

## Notes

<sup>1</sup>See Eydeland and Wolyniec (2003). The approximation derived in Deng *et al.* (2007, 2008) is not based on dimension reduction. Nevertheless, it has the problem that the prices still depend on a subjective choice for the strikes of the single-asset implied volatilities.

<sup>2</sup>To see why, consider a call spread option with zero strike as an example. We can buy a call option on asset 1 with strike  $K_1$  and short a call option on asset 2 with strike  $K_2$  where  $K_1 - K_2$  is equal to the expected pay-off of the spread option and the individual strikes are the expected maturity values of  $S_{1,t}$  and  $S_{2,t}$ . So the pay-off will be  $K_1 - K_2$  if  $S_{1,T} \geq S_{2,T}$ .

- <sup>3</sup>In the framework of Alexander and Venkatramanan (2011), the exchange options and the vanilla calls and puts in the exchange options need not be traded. Hence, we are free to choose the strikes of the vanilla options as we please. However, it should be noted that if their strikes are very far outside the normal range for traded options, the spread option price will be subject to model risk arising from the method used to extrapolate the volatility smile.
- <sup>4</sup>Hedge portfolios of single-asset call and put options are constructed by picking the coefficients of the corresponding single-asset Greeks on the right-hand side of Equation (12) and adding them together. This implies that the CEO model hedge ratios are indeed consistent with the endogenous single-asset option strikes given in Theorem 3.1.
- <sup>5</sup>Recall that the approximation error will be smallest when we choose the vanilla options in the exchange options to be as deep ITM as possible.
- <sup>6</sup>For instance, when implied volatilities can be closely fitted by a cubic function, Equation (14) reduces to a cubic equation, whose roots can be found very easily.
- <sup>7</sup>See McKean (1965), Carr *et al.* (1992), Kim (1990) and Jacka (1991) for single-asset American option examples.
- <sup>8</sup>Similar work has been carried out by Carr *et al.* (1992) and Jacka (1991) who derived different representations to the EEP. Ju (1998) and Huang *et al.* (1996) proposed efficient techniques to compute the exercise boundary; the former approximates the exercise boundary using a multi-piece exponential function that can be found by solving a set of equations analytically and the latter implement a four-point Richardson extrapolation scheme.
- <sup>9</sup>Errors are reported as a percentage of the option price. For comparison, the RMSE was 9.3% when we used the constant ATM volatility to determine  $\sigma_1$  and  $\sigma_2$ . Results for other strike conventions are not reported for reasons of space but are available on request.
- <sup>10</sup>The average values of the strike  $K_1 = mK$  of the corresponding vanilla call and put options on gasoline are given in parentheses.
- <sup>11</sup>The deltas shown are those for put spread options. The call delta with respect to gasoline increases with strike because the call spread option price increases as gasoline prices increase, and the delta with respect to crude oil decreases with strike because the call spread option price decreases with an increase in crude oil prices. As the options move deeper ITM the absolute value of both CEO deltas approach 1, and as the options move deeper OTM, they approach 0.

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