Generalized Beta-Generated Distributions

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Abstract

This paper introduces a new class of generalized beta-generated distributions that have very flexible shapes and tractable properties. Their quantiles and moments have a simple closed form and they are maximum entropy distributions under three simple conditions. Two special cases are the classical beta-generated and the Kumaraswamy-generated distributions. An attractive feature of generalized beta-normal distributions is that the three generalized beta parameters afford greater control over the weights in both tails and in the centre of the generated distribution, compared with the classical beta-normal distribution.

Key Words: generalized beta; Kumaraswamy minimax; generated distributions; maximum entropy

JEL Codes: C16, G1

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1 Motivation

The classical beta distribution $\mathcal{B}(p,q)$ may be characterized by its density function:

$$f_{\mathcal{B}}(u;p,q) = B(p,q)^{-1}u^{p-1}(1-u)^{q-1}, \qquad 0 \le u \le 1, \tag{1}$$

where $B(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ represent the beta function, $\Gamma(\cdot)$ the gamma function, and the two parameters p and q are such that $p,q > 0$. Although it has only two parameters, the beta density accommodates a very wide variety of shapes including (for $p = q = 1$) the standard uniform distribution $U[0,1]$. The beta density is symmetric iff $p = q$, is unimodal when $p,q > 1$ and 'U' shaped when $p,q < 1$. It has positive skew when $p < q$ and negative skew when $p > q$.

The first distribution of the beta-generated class was the beta-normal distribution introduced by Eugene, Lee and Famoye (2002). Denote the standard normal distribution and density functions by $\Phi(.)$ and $\phi(.)$ respectively, and let $X = \Phi^{-1}(U)$ with $U \sim \mathcal{B}(p,q)$. Then X has a beta-normal distribution $\mathcal{BN}(p,q;0,1)$ with density function:

$$f_{\mathcal{BN}}(x;p,q,0,1) = B(p,q)^{-1}\phi(x)[\Phi(x)]^{p-1}[1-\Phi(x)]^{q-1}, \qquad -\infty < x < \infty. \tag{2}$$

If $X \sim \mathcal{BN}(p,q;0,1)$ then $Y = \sigma X + \mu \sim \mathcal{BN}(p,q;\mu,\sigma)$ has the non-standard beta-normal distribution with $N(\mu,\sigma^2)$ parent. That is, Y has density function

$$f_{\mathcal{BN}}(y;p,q,\mu,\sigma) = \sigma^{-1}B(p,q)^{-1}a\phi(\frac{y-\mu}{\sigma})\Phi(\frac{y-\mu}{\sigma})^{p-1}[1-\Phi(\frac{y-\mu}{\sigma})]^{q-1}. \tag{3}$$

The beta-normal density is symmetric iff $p = q$. It has negative skewness when $p < q$ and positive skewness when $p > q$. When $p = q > 1$ the beta-normal distribution has positive excess kurtosis and when $p = q < 1$ it has negative excess kurtosis, as demonstrated by Eugene, Lee and Famoye (2002).

However, using the beta-normal distribution has very limited values for both skewness and kurtosis coefficients. Eugene, Lee and Famoye (2002) tabulate the mean, variance, skewness and kurtosis of $\mathcal{BN}(p,q;0,1)$ for some particular values of p and q between 0.05 and 100. The skewness always lies in the interval $(-1,1)$ and the largest kurtosis value found is 4.1825 (for $p = 100$ and $q = 0.1$ and vice versa). These results also apply to general beta-normal distributions as their skewness and kurtosis is the same as for standard beta-normal distributions with the same p and q. They show that the only way to gain even a modest degree of excess kurtosis is to skew the distribution as far as possible.

Changing the parent distribution so that it is no longer normal offers more flexibility. Replacing Φ in (4) by any other parent distribution F yields the general class of beta-generated distributions introduction by Jones (2004). These may be characterized by their density function

$$f_{\mathcal{BG}} = B(p,q)^{-1} f(x)[F(x)]^{p-1}[1-F(x)]^{q-1}, \qquad x \in \mathcal{I}, \tag{4}$$

where $F(x)$ is the parent distribution function and $f(x)$ is its density. Jones (2004) concentrates on the cases where F is symmetric about zero with no free parameters other than location and scale and where $\mathcal{I}$ is the whole real line. With a symmetric parent p and q control the degree of skewness introduced, with sign equal to the sign of $p - q$. Tail weight is also governed by p and q: when $p = q = 1$ tails have the same weight as the parent and tails become lighter as p increases and heavier as q decreases.

Beta-generated distributions with more general parents have been studied by Nadarajah and Kotz (2004, 2005), Akinsete, Famoye and Leeb (2008), Zografos and Balakrishnan (2009) and Barreto-Souza, Cordeiro and Simas (2010). Jones and Larsen (2004) and Arnold, Castillo and Sarabia (2006) introduce the multivariate beta-generated class. Some practical applications of these distributions have been considered: Jones and Larsen (2004) fit skewed t and log F distributions, which Jones (2004) shows are special cases of beta-generated distributions, to temperature data; Akinsete, Famoye and Leeb (2008) apply the beta-Pareto distribution to flood data; Razzaghi (2009) applies the beta-normal distribution to dose-response modeling.

The shapes of beta-generated distributions are more flexible than the beta-normal. However, they seem unable to add much to the kurtosis of the parent unless an extreme skew is introduced. To demonstrate this, Table 1 gives the kurtosis of a beta-Student t distribution with $p = q$ and where the parent Student t

	Degrees of Freedom		
p = q	**Normal (∞)**	**10**	**6**
0.1	2.35200	–	–
1	3	4	6
2	3.03473	3.39126	3.75263
2.5	3.03417	3.30010	3.54652
5	3.02326	3.13801	3.22959
7	3.01776	3.09621	3.15660
7.5	3.01675	3.08943	3.14506
10	3.01300	3.06611	3.10599
100	3.00141	3.00635	3.00990
1000	3.00014	3.00063	3.00098
10000	3.00001	3.00006	3.00010
1000000	3.00000	3.00001	3.00001

disrtibution has 6, 10 and ∞ degrees of freedom. With the normal generator, the values shown in the column headed 'normal' display a very modest degree of excess kurtosis, and only when $p = q > 1$. With the Student t generators having 10 and 6 degrees of freedom, the maximum kurtosis is obtained when $p = q = 1$ and when $p = q > 1$ the kurtosis of the generated distribution declines rapidly to 3 as p and q increase.[1]

Classical beta-generated distributions are limited in two respects. Firstly, for many choices of parent the computations of quantiles and moments can become rather complex. And secondly, the classical beta distribution has only two parameters and so it can add only a limited structure to the generated distribution. For instance, a beta-generated distribution may have problems to capture the behaviour of random variables with symmetric but highly leptokurtic distributions. Whilst the beta parameters offer explicit control over skewness when the parent is symmetric, they have less control over higher moments such as kurtosis. Jones (2004) describes the tail weight of beta-generated distribution when the parent has exponential, power and normal tails, but it is not clear how his results translate into kurtosis.[2]

[1]When $p = q < 1$ the numerical integration of the fourth order moment fails, except in the normal case (i.e. infinite degrees of freedom), indicated by the '–' in the table.

[2]In the first case, the exponential tails remain. In the power tail case the tail weight is proportional to $|x|^{pq-1}$ and in the normal case the tails are proportional to $|x|^{1-a}\exp(-\frac{ax^2}{2})$ with $a = q$

This paper takes a different approach to much of the literature so far: rather than retaining a classical beta generator and considering more flexible parent distributions than the normal, we propose the use of a more flexible generator distribution. In particular, our generator is the generalized beta distribution of the first kind, denoted $\mathcal{GB}(a, p, q)$. Special cases of generalized beta-generated (GBG) distributions include the classical beta-generated and the Kumaraswamy generated distributions. We show that the generalized beta-normal (GBN) distribution and the more general class of GBG distributions have tractable properties. In particular, they have closed-form quantiles and moments and they are maximum entropy distributions under three fairly general constraints.

2 The GBG distribution and its properties

The generalized beta distribution of the first kind was introduced by McDonald (1984). It may be characterized by its density function

$$f_{\mathcal{GB}}(u; a, p, q) = B(p, q)^{-1}[au^{ap-1}(1 - u^a)^{q-1}], \quad 0 < u < 1. \tag{5}$$

Two important special cases are the classical beta distribution ($a = 1$), and the Kumaraswamy distribution ($p = 1$). The distribution of Kumaraswamy (1980) is commonly termed the 'minimax' distribution. Jones (2009) advocates its tractability, especially in simulations because its quantile function takes a simple form, and its pedagogical appeal relative to the classical beta distribution. Indeed, it has more desirable generator properties than the classical beta distribution when used in conjunction with a normal parent, as we shall see below.

The beta-generated distributions described in Section 1 are now extended to a more general class of generalized beta-generated (GBG) distributions. Given a parent distribution $F(x)$, $x \in \mathcal{I}$ with density $f(x)$, the GBG density takes the form

$$f_{\mathcal{GBG}}(x; a, p, q) = B(p, q)^{-1} f(x)[aF(x)^{ap-1}(1 - F(x)^a)^{q-1}], \ x \in \mathcal{I}. \tag{6}$$

Two important special cases are the beta-generated distribution ($a = 1$), and the

for the upper tail and $a = p$ for the lower tail.

Kumaraswamy generated distribution ($p = 1$). It follows immediately from (6) that the GBG with parent $F(x)$ is a standard beta-generated distribution with parent $F(x)^a$. This simple transformation facilitates the computatation of many of its properties.

2.1 Cumulative distribution and quantile functions

Let X be a GBG distribution with pdf (6), which will be represented by $X \sim \mathcal{GBG}(a, p, q, F)$. The random variable X admits the simple stochastic representation,

$$X = F^{-1}(U^{1/a}), \tag{7}$$

where $U \sim \mathcal{B}(p, q)$. Using the transformation (7), the cumulative distribution function of (6) may be written:

$$F_{\mathcal{GBG}}(x; a, p, q) = I(F(x)^a; p, q), \tag{8}$$

where $I(x; p, q)$ denotes the incomplete beta ratio function.

The quantile function of a GBG distribution is given by

$$Q_{\mathcal{GBG}}(x; a, p, q, F) = F^{-1}[\left[I^{-1}(x; p, q)\right]^{1/a}], \tag{9}$$

where $I^{-1}(u; p, q)$ represents the inverse of the incomplete beta ratio function.

We remark than when either p or q is an integer (8) may be written in series form. In particular, if q is an integer:

$$F_{\mathcal{GBG}}(x; a, p, q) = \sum_{r=0}^{q-1} \binom{p+q-1}{r} F(x)^{a(p+q-r-1)}[1 - F(x)^a]^r.$$

Alternatively, if p is an integer then

$$F_{\mathcal{GBG}}(x; a, p, q) = 1 - \sum_{r=0}^{p-1} \binom{p+q-r}{r} F(x)^{ar}[1 - F(x)^a]^{p+q-r-1}.$$

2.2 Moments

The moments of (6) may be obtained using representation (7).[3] If $X \sim \mathcal{GBG}(a, p, q, F)$ and $U \sim \mathcal{B}(p, q)$ we have

$$\begin{aligned} E[X^r] &= E\{[F^{-1}(U^{1/a})]^r\} \\ &= B(p,q)^{-1}\int_0^1 [F^{-1}(x)]^r \; ax^{ap-1}(1-x^a)^{q-1}dx. \end{aligned} \tag{10}$$

A simple approximation to integral (10) can be obtained upon expanding $F^{-1}(x)$ in a Taylor series around the point $E(X_F) = \mu_F$:

$$E(X^r) \approx \sum_{k=0}^{r}\binom{r}{k}[F^{-1}(\mu_F)]^{r-k}[F^{-1(1)}(\mu_F)]^k \sum_{i=0}^{k}(-1)^i\binom{k}{i}\mu_F^i \frac{B(p+\frac{k-i}{a}, q)}{B(p,q)},$$

where $F^{-1(1)}(x) = \frac{d}{dx}F^{-1}(x) = [f(F^{-1}(x))]^{-1}$. Moreover,

$$E[F(X)^r] = \frac{B(p+\frac{r}{a}, q)}{B(p,q)}, \tag{11}$$

$$E[[1-F(X)^a]^s] = \frac{B(p, q+s)}{B(p,q)}, \tag{12}$$

$$E[F(X)^r[1-F(X)^a]^s] = \frac{B(p+\frac{r}{a}, q+s)}{B(p,q)}. \tag{13}$$

The relationships (11) – (13) may be used to obtain initial estimators for the parameters p and q assuming a is known and F is given. In particular, considering (11) with $r = a$ and (13) with $r = a$ and $s = 1$ we have:

$$E[F(X)^a] = \frac{p}{p+q}, \tag{14}$$

$$E[F(X)^a[1-F(X)^a]] = \frac{pq}{(p+q)(p+q+1)}. \tag{15}$$

[3] Other kinds of moments related to the L-moments of Hosking (1990) may also be obtained in closed form.

Solving (14) and (15) for p and q yields

$$p = \frac{uv}{u(1-u)-v}, \tag{16}$$

$$q = \frac{v(1-u)}{u(1-u)-v}, \tag{17}$$

where $u = E[F(X)^a]$ and $v = E[F(X)^a[1 - F(X)^a]]$.

2.3 Some special cases

Finally, we include some examples of generalized beta-generated distributions.

(a) *Generalized beta-normal distributions*
Taking F to be a standard normal distribution we define the generalized beta-normal distribution by its density function

$$f(x; a, p, q) = \frac{a\phi(x)\Phi(x)^{ap-1}[1 - \Phi(x)^a]^{q-1}}{B(p, q)}, \quad -\infty < x < \infty, \tag{18}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ represent the density and distribution functions of the standard normal distribution. If we set $a = 1$ in (18) we obtain the beta-normal distribution proposed by Eugene et al. (2002). When a, p and q are integers, the moments of (18) can be obtained in terms of the Lauricella function of type A (Exton, 1978) using the methodology proposed by Nadarajah (2008).

(b) *Generalized beta-lognormal distributions*
Taking F to be a standard lognormal distribution with distribution function $F(x) = \Phi(\log x)$ we characterize the generalized beta-lognormal distribution by its distribution

$$f(x; a, p, q) = \frac{a\phi(\log x)\Phi(\log x)^{ap-1}[1 - \Phi(\log x)^a]^{q-1}}{xB(p, q)}, \quad 0 < x < \infty. \tag{19}$$

Again, when a, p and q are integers, the moments of (19) can be obtain as a finite sum of the Lauricella function of type A.

(c) *Generalized beta-skewed-t distributions*

Let X be a scaled Student t distribution on 2 degrees of freedom with scaled factor $\sqrt{\lambda/2}$ and distribution

$$F(x) = \frac{1}{2}\left(1 + \frac{x}{\sqrt{\lambda + x^2}}\right), \quad -\infty < x < \infty, \tag{20}$$

where $\lambda = p + q$. Taking (20) as F in (6) we obtain a class of density functions defined as follows:

$$f(x; a, p, q) = \frac{\lambda a}{2^{ap} B(p,q)} \frac{1}{(\lambda + x^2)^{3/2}} \left(1 + \frac{x}{\sqrt{\lambda + x^2}}\right)^{ap-1} \left[1 - \frac{1}{2^a}\left(1 + \frac{x}{\sqrt{\lambda + x^2}}\right)^a\right]^{q-1}, \tag{21}$$

where $-\infty < x < \infty$ and $\lambda = p + q$. Two special cases deserve our attention:

- When $a = 1$ and $p = q$ we obtain a Student t distribution with $2p$ degrees of freedom.
- When $a = 1$ we obtain the skewed $t(p, q)$ distribution proposed by Jones and Faddy (2003).

(d) *Generalized beta-Laplace distributions*

The generalized beta-Laplace distribution is defined by expression (6) where now F is a scaled Laplace distribution, having distribution function

$$F(x) = \begin{cases} \frac{1}{2}\exp(x/\lambda), & x < 0, \\ 1 - \frac{1}{2}\exp(-x/\lambda), & x > 0. \end{cases} \tag{22}$$

where $\lambda > 0$. This distribution is a new class of skewed Laplace distributions.

(e) *Translation of location and scale*

Let X be a GBG random variable with density

$$f_{\mathcal{GBG}}(x; a, p, q) = B(p,q)^{-1} a f(x) F(x)^{ap-1} [1 - F(x)^a]^{q-1},$$

and assume $E(X) = \mu_F$ and $var(X) = \sigma_F^2$ exist and are finite. If we define $Y = \mu + \sigma X$, then Y has density

$$
\begin{aligned}
f(y; a, p, q, \mu, \sigma) &= \sigma^{-1} f_{\mathcal{GBG}}(\frac{y-\mu}{\sigma}; a, p, q) \\
&= \sigma^{-1} B(p,q)^{-1} a f(\frac{y-\mu}{\sigma}) F(\frac{y-\mu}{\sigma})^{ap-1}[1 - F(\frac{y-\mu}{\sigma})^a]^{q-1},
\end{aligned}
$$

and $E(Y) = \mu + \sigma\mu_F$ and $var(Y) = \sigma^2\sigma_F^2$.

3 Entropy

Shannon (1948) defined the *entropy* of a probability density function $g(x)$ as

$$
H(g) = \mathrm{E}_g[\log g(x)] = -\int g(x) \log g(x) dx. \tag{23}
$$

The Shannon entropy (henceforth called simply entropy) is a measure of the uncertainty in a probability distribution and its negative is a measure of information. Ebrahimi, Maasoumi and Soofi (1999) show that there is no universal relationship between variance and entropy and where their orderings differ entropy is the superior measure of information.

Proposition 1 *The entropy of the generalized beta distribution with density (5) is*

$$
\begin{aligned}
-E_{\mathcal{GB}}[\log f_{\mathcal{GB}}(x)] &= \log B(p,q) - \log a + a^{-1}(a-1)\zeta(p,q) \\
&\quad +(p-1)\zeta(p,q) + (q-1)\zeta(q,p). \qquad (24)
\end{aligned}
$$

where $\zeta(p,q) = \psi(p+q) - \psi(p)$ *and* ψ *represents the digamma function.*

Proof: Note that (5) may be written:

$$
f_{\mathcal{GB}}(x; a, p, q) = B(p,q)^{-1} g(x)[G(x)]^{p-1}[1 - G(x)]^{q-1}, \tag{25}
$$

where $G(x) = x^a, \ \ 0 < x < 1$ and $g(x) = G'(x)$. Hence the generalized beta is a classical beta-generated distribution with parent $G(x) = x^a$. Hence

$$
-\mathrm{E}_{\mathcal{GB}}[\log f_{\mathcal{GB}}(x)] = \log B(p,q) - \mathrm{E}_{f_{\mathcal{GB}}}[\log g(x)]
$$

$$-(p-1)\mathrm{E}_{f_{\mathcal{GB}}}[\log G(x)] - (q-1)\mathrm{E}_{f_{\mathcal{GB}}}[\log(1-G(x))].$$

Now (24) follows on substituting

$$\begin{aligned}
-\mathrm{E}_{\mathcal{GB}}[\log G(x)] &= \zeta(p,q), \\
-\mathrm{E}_{\mathcal{GB}}[\log(1-G(x))] &= \zeta(q,p), \\
\mathrm{E}_{\mathcal{GB}}[\log g(x)] &= \int_0^1 (\log a + (a-1)\log x) f_{\mathcal{GB}}(x) dx \\
&= \log a - a^{-1}(a-1)\zeta(p,q). \qquad \blacksquare
\end{aligned}$$

Corollary 1 *The entropy of the classical beta distribution $\mathcal{B}$ is:*

$$-E_{\mathcal{B}}[\log f_{\mathcal{B}}(x)] = \log B(p,q) + (p-1)\zeta(p,q) + (q-1)\zeta(q,p). \tag{26}$$

The entropy of the Kumaraswamy distribution $\mathcal{K}$ is:

$$-E_{\mathcal{K}}[\log f_{\mathcal{K}}(x)] = -\log(aq) + a^{-1}(a-1)\zeta(1,q) + (q-1)\zeta(q,1). \tag{27}$$

We remark that (26) is well known, see Nadarajah and Zografos (2003), and (27) follows from (24) on noting that $B(1,q) = q^{-1}$.

The maximum entropy density is the function $f(x)$ that maximizes $H(g)$, subject to a set of conditions on $g(x)$ which capture the testable information that is available to the analyst.[4] The criterion here is to be a vague as possible (i.e. to maximize uncertainty) given the constriants imposed by the testable information, so that the maximum entropy distribution (MED) represents no more (and no less) than this information.

Zografos and Balakrishnan (2009) prove that standard beta-generated distributions are MED under three constraints, two relating only to the beta generator and one relating to the parent. Using their results, we now compute the entropy of GBG distribution and the conditions under which it is the MED.

[4]A piece of information is *testable* if it can be determined whether $g(x)$ is consistent with it. One of piece of information is always a normalization condition.

Proposition 2 *The GBG density (6) with parent distribution F statisfies,*

$$
\begin{aligned}
-E_{\mathcal{GBG}}[\log F(X)^a] &= \zeta(p,q), && (28)\\
-E_{\mathcal{GBG}}[1-\log F(X)^a] &= \zeta(q,p), && (29)\\
E_{\mathcal{GBG}}[\log f(X)] - E_U[\log f(F^{-1}(U^{1/a}))] &= a^{-1}(a-1)^2\zeta(p,q) && (30)
\end{aligned}
$$

where $f(x) = F'(x)$ and $U \sim \mathcal{B}(p,q)$. Furthermore, the GBG has the maximum entropy of all distributions satisfying the information constraints (28) – (30) and its entropy is:

$$
\begin{aligned}
-E_{\mathcal{GBG}}[\log f_{\mathcal{GBG}}(x)] &= \log B(p,q) - \log a + a^{-1}(a-1)\zeta(p,q)\\
&\quad +(p-1)\zeta(p,q) + (q-1)\zeta(q,p) - E_U[\log f(F^{-1}(U^{1/a}))] \quad (31)
\end{aligned}
$$

Proof: Since the generalized beta-generated distribution with parent $F(x)$ is a beta-generated distribution with parent $G(x) = F(x)^a$, we may use Lemma 1 of Zografos and Balakrishnan (2009). The conditions (28) and (29) follow immediately from the first two conditions that lemma and the third condition is

$$
\mathrm{E}_{\mathcal{BG}}[\log g(X)] = \mathrm{E}_U[\log g(G^{-1}(U))], \qquad U \sim \mathcal{B}(p,q), \qquad (32)
$$

where $g(x) = G'(x) = aF(x)^{a-1}f(x)$. Now $G^{-1}(U) = F^{-1}(U^{1/a})$ and

$$
\log g(X) = \log a + (a-1)\log F(X) + \log f(X),
$$

so

$$
\begin{aligned}
\mathrm{E}_{\mathcal{B}}[\log g(G^{-1}(U))] &= \log a + (a-1)\mathrm{E}_{\mathcal{B}}[\log U^{1/a}] + \mathrm{E}_{\mathcal{B}}[\log f(F^{-1}(U^{1/a}))],\\
&= \log a - a^{-1}(a-1)\zeta(p,q) + \mathrm{E}_{\mathcal{B}}[\log f(F^{-1}(U^{1/a}))], \quad (33)
\end{aligned}
$$

since

$$
\mathrm{E}_{\mathcal{B}}[\log U] = B(p,q)^{-1}\int_0^1 u^{p-1}(1-u^p)^{q-1}\log u \; du = -\zeta(p,q).
$$

By Lemma 1 of Zografos and Balakrishnan (2009), $\mathrm{E}_{\mathcal{BG}}[\log F(X)] = -\zeta(p,q)$. Hence (32) becomes:

$$(a-1)\zeta(p,q) - \mathrm{E}_{\mathcal{GBG}}[\log f(X)] = a^{-1}(a-1)\zeta(p,q) - \mathrm{E}_{\mathcal{B}}[\log f(F^{-1}(U^{1/a}))]. \quad (34)$$

which may be rewritten as (30). Finally, (31) follows from (33) and corollary 1 of Zografos and Balakrishnan (2009) and their proposition 1 proves the maximum entropy property. ∎

Setting $a = 1$ in (28) – (31) yields the information constraints and entropy for the beta-generated distribution that were derived directly in Zografos and Balakrishnan (2009). Setting $p = 1$ in (28) – (31) yields the information constraints for the Kumaraswamy-generated distribution $\mathcal{K}$ to have maximum entropy, and this entropy is given by:

$$\begin{aligned} -\mathrm{E}_{\mathcal{KG}}[\log f_{KG}(x)] &= -\log(aq) + a^{-1}(a-1)\zeta(1,q) \\ &\quad +(q-1)\zeta(q,1) - \mathrm{E}_U[\log f(F^{-1}(U^{1/a}))]. \end{aligned} \quad (35)$$

We remark that the GBG entropy is the sum of the entropy of the generalized beta generator (24), which is independent of the parent, and another term $-\mathrm{E}_U[\log f(F^{-1}(U^{1/a}))]$ that is related to the entropy of the parent. Furthermore, the constraints (28) and (29) reflect information only about the generalized beta generator. From

$$\mathrm{E}_{\mathcal{GBG}}[\log F(X)^a] = \mathrm{E}_{\mathcal{BG}}[\log G(x)] = \mathrm{E}_{\mathcal{B}}[\log U],$$

$$\mathrm{E}_{\mathcal{GBG}}[\log F(X)] = \mathrm{E}_{\mathcal{BG}}[\log(1-G(x))] = \mathrm{E}_{\mathcal{B}}[\log(1-U)],$$

it follows that (28) is related to the information in the left tail and (29) is related to information in the right tail. Note that (24) takes its maximum value of zero when $a = p = q = 1$; otherwise, the structure in the generator adds information to the GBG distribution.

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